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Academician of the Academy of Sciences of the Armenian SSR M. M. Dzhrbashyan and R. M. Martirosyan

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Abstract

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THE MOMENT PROBLEM AND BIORTHOGONALIZATION OF KERNELS

In a note by one of the authors (^{1}), questions were considered concerning quasi-isometric mappings of Hilbert spaces of functions L^2_σ, which were further developed in our work (^{2}), where the concepts of Hilbert and Bessel kernels were introduced and some of their properties were established, these being natural continuous analogues of results of N. K. Bari. In the present note, which is a continuation of the preceding investigations, the continuous moment problem is posed in general form, and the results obtained are applied to the problem of biorthogonalization of a certain class of kernels $\widetilde{K}(\xi, x)$. Further, the question is clarified as to whether a kernel $\widetilde{K}(\xi, x)$, "quadratically close" to a certain Riesz kernel (^{2}), is a Riesz kernel. This result may be regarded as a natural continuous analogue of N. K. Bari' s theorem (^{3}) on minimal systems quadratically close to bases in Hilbert spaces.

Let $\sigma_k(x)$ $(k = 1, 2)$ be a nondecreasing function, defined and right-continuous on the interval (a_k, b_k), where $-\infty \leq a_k \leq b_k \leq \infty$, having bounded variation on every segment $[\alpha, \beta] \subset (a_k, b_k)$. Denote by $H_k = L^2_{\sigma_k}(a_k, b_k)$ the Hilbert space of all σ_k-measurable functions on (a_k, b_k) that are square-summable with respect to σ_k.

Let $\widetilde{K}(\xi, x)$ be a function of two variables $\xi \in (a_2, b_2)$ and $x \in (a_1, b_1)$, and suppose that for all $\xi \in (a_2, b_2)$ the condition $\widetilde{K}(\xi, x) \in H_1$ is satisfied. Suppose further that for every point of continuity ξ_0 of the function $\sigma_2(\xi)$ there exist $\Delta_{\xi_0} > 0$ and positive constants $c(\xi_0)$ and $\alpha(\xi_0)$ such that for all δ, $|\delta| < \Delta_{\xi_0}$, the inequality

$$\int_{a_1}^{b_1} \left|\widetilde{K}(\xi_0 + \delta, x) - \widetilde{K}(\xi_0, x)\right|^2 d\sigma_1(x) \leq c(\xi_0) \left|\sigma_2(\xi_0 + \delta) - \sigma_2(\xi_0)\right|^{\alpha(\xi_0)}. \quad (1)$$

is satisfied.

Any function $\widetilde{K}(\xi, x)$ satisfying the listed conditions will, for brevity, be called a C-kernel.

Theorem 1. *Let $\widetilde{K}(\xi, x)$ be a C-kernel, and let $\mu(\xi)$ be continuous at all points of continuity of the function $\sigma_2(\xi)$, and suppose that for some $\gamma > 0$ the inequality*

$$\left|\int_{a_2}^{b_2} \mu(\xi) g(\xi)\, d\sigma_2(\xi)\right| \leq \gamma \left(\int_{a_1}^{b_1} \left|\int_{a_2}^{b_2} \widetilde{K}(\xi, x) g(\xi)\, d\sigma_2(\xi)\right|^2 d\sigma_1(x)\right)^{1/2} \tag{2}$$

holds for all step functions $g(\xi)$ that vanish outside finite intervals lying entirely inside (a_2, b_2). Then there exists a function $f(x) \in H_1$, $\|f\|_{\sigma_1} \leq \gamma$, such that for all $\xi \in (a_2, b_2)$ not lying inside intervals of constancy of the function $\sigma_2(\xi)$,

$$\mu(\xi) = \int_{a_1}^{b_1} \widetilde{K}(\xi, x) f(x)\, d\sigma_1(x). \tag{3}$$

The proof is based on the fact that, when the conditions of the theorem are fulfilled, the inequality

$$\left|\sum_{k=1}^{n} \lambda_k \mu(\xi_k)\right| \leq \gamma \left\|\sum_{k=1}^{n} \lambda_k \widetilde{K}(\xi_k, x)\right\|_{\sigma_1} \tag{4}$$

holds for any n and any choice of the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, with the same constant γ as in the condition of the theorem. Here ξ_k $(k = 1, 2, \dots)$ is the totality of all points of discontinuity of the function $\sigma_2(\xi)$, supplemented by all such rational points from (a_2, b_2) for which no neighborhood $(\xi_k - \delta, \xi_k + \delta)$ has zero σ_2-measure, if ξ_k is a point of continuity of the function $\sigma_2(\xi)$.

Remark. If for all $x \in (a_1, b_1)$ the function $K(\xi, x)$ is σ_2-measurable on (a_2, b_2), and if on every finite interval lying entirely inside (a_2, b_2) the function

$$\int_{a_1}^{b_1} |\widetilde{K}(\xi, x)|^2\, d\sigma_1(x)$$

is bounded, then, as is easy to see, for the solvability of the moment problem, i.e., for the fulfillment of (3), condition (2) is necessary. In this case one may take $\gamma = \|f\|_{\sigma_1}$.

We now introduce the following definition. We shall say that a kernel $\widetilde{K}(\xi, x)$ **admits biorthogonalization in H_1** if there exists a function $\widetilde{K}_*(\xi, x)$ such that, for any choice of $\xi, \eta \in (a_2, b_2)$, the equality

$$\int_{a_2}^{b_2} e_\xi(x) e_\eta(x)\, d\sigma_2(x) = \int_{a_1}^{b_1} \widetilde{K}(\xi, x) \overline{\widetilde{K}_*(\eta, x)}\, d\sigma_1(x),$$

holds, where $e_\xi(x)$ and $-e_\xi(x)$ are the characteristic functions of the intervals $[0,\xi)$ and $[\xi,0)$, respectively.

If the kernel $\widetilde{K}(\xi,x)$ is complete in H_1, i.e., if the linear span of the functions $\widetilde{K}(\xi,x)$, considered for all $\xi \in (a_2,b_2)$, is dense in H_1, then, clearly, $\widetilde{K}_*(\eta,x)$ is determined uniquely and in this case is called the kernel **adjoint to** $\widetilde{K}(\xi,x)$.

We consider it not superfluous to note that if the adjoint kernel $\widetilde{K}_*(\xi,x)$ is also complete in H_1, then, using the results of paper (1), one can show that condition (1), imposed on the kernel $\widetilde{K}(\xi,x)$ in Theorem 1, is, generally speaking, necessary, with $\alpha(\xi) \equiv 1$.

Denote

$$\mu(\xi,\eta) = \int_{a_2}^{b_2} e_\xi(x) e_\eta(x)\, d\sigma_2(x).$$

Then the preceding theorem immediately implies:

Theorem 2. Suppose that the C-kernel $\widetilde{K}(\xi,x)$ is such that, for all $\eta \in (a_2,b_2)$, the inequality

$$\left|\int_{a_2}^{b_2} \mu(\xi,\eta) g(\xi)\, d\sigma_2(\xi)\right| \le \gamma(\eta) \left(\int_{a_1}^{b_1} \left|\int_{a_2}^{b_2} \widetilde{K}(\xi,x) g(\xi)\, d\sigma_2(\xi)\right|^2 d\sigma_1(x)\right)^{1/2}$$

is satisfied for all step functions $g(\xi)$ that vanish outside finite intervals lying entirely inside (a_2,b_2). Then the kernel $\widetilde{K}(\xi,x)$ admits biorthogonalization in H_1.

Before formulating the next theorem, let us recall that $K(\xi,x)$ is called a (2) **kernel of an isometric operator** if it is complete in H_1 and, for all $\xi,\eta \in (a_2,b_2)$,

$$\int_{a_1}^{b_1} K(\xi,x)\overline{K(\eta,x)}\, d\sigma_1(x) = \int_{a_2}^{b_2} e_\xi(x) e_\eta(x)\, d\sigma_2(x). \tag{5}$$

At the same time, as was shown in (2), there exists an isometric operator V mapping the whole space H_1 onto the whole space H_2, such that for every $f(x) \in H_1$ the equality

$$\int_{a_2}^{b_2} Vf(x) e_\xi(x)\, d\sigma_2(x) = \int_{a_1}^{b_1} f(x)\overline{K(\xi,x)}\, d\sigma_1(x). \tag{6}$$

Theorem 3. Let some kernel $\widetilde{K}(\xi, x)$ be complete in H_1, and let the moment problem

$$\int_{a_2}^{b_2} g(x)e_\xi(x)\,d\sigma_2(x) = \int_{a_1}^{b_1} \widetilde{K}(\xi, x)f(x)\,d\sigma_1(x) \tag{7}$$

be solvable in H_1 for every $g(x) \in H_2$. Then, whatever the kernel $K(\xi, x)$ of an arbitrary isometric operator mapping H_1 onto H_2, there exists such a bounded linear operator T, defined on H_1, that the kernel $\widetilde{K}_*(\xi, x)$ adjoint to $\widetilde{K}(\xi, x)$ is determined by the formula $\widetilde{K}_*(\xi, x) = TK(\xi, x)$. At the same time also $K(\xi, x) = T^*\widetilde{K}(\xi, x)$.

For the formulation of the next theorem it is necessary to recall the definition of the so-called Riesz kernels, first introduced into consideration in our note (^{2}).

A kernel $\widetilde{K}(\xi, x)$ complete in H_1 ($\xi \in (a_2, b_2)$, $x \in (a_1, b_1)$) is called a **Riesz kernel** if there exist two such kernels $\widetilde{K}_*(\xi, x)$ ($\xi \in (a_2, b_2)$, $x \in (a_1, b_1)$) and $\widetilde{H}_*(\xi, x)$ ($\xi \in (a_1, b_1)$, $x \in (a_2, b_2)$), complete respectively in H_1 and H_2 and connected with $\widetilde{K}(\xi, x)$ by the relations

$$\int_{a_2}^{b_2} e_\xi(x)e_\eta(x)\,d\sigma_2(x) = \int_{a_1}^{b_1} \widetilde{K}(\xi, x)\overline{\widetilde{K}_*(\eta, x)}\,d\sigma_1(x),$$

$$\int_{a_1}^{b_1} \widetilde{K}(\eta, x)e_\xi(x)\,d\sigma_1(x) = \int_{a_2}^{b_2} \overline{\widetilde{H}_*(\xi, x)}e_\eta(x)\,d\sigma_2(x), \tag{8}$$

so that to any function $f(x) \in H_1$ there corresponds some function $g^*(x) \in H_2$ such that for all $\xi \in (a_1, b_1)$

$$\int_{a_1}^{b_1} f(x)e_\xi(x)\,d\sigma_1(x) = \int_{a_2}^{b_2} \overline{\widetilde{H}_*(\xi, x)}g^*(x)\,d\sigma_2(x), \tag{9}$$

and moreover for all $\xi \in (a_2, b_2)$

$$\int_{a_2}^{b_2} g^*(x)e_\xi(x)\,d\sigma_2(x) = \int_{a_1}^{b_1} f(x)\overline{\widetilde{K}_*(\xi, x)}\,d\sigma_1(x). \tag{10}$$

From these conditions, on the basis of the results of (^{1}), one can conclude that the kernel $\widetilde{H}_*(\xi, x)$ admits biorthogonalization, i.e. for all $\xi, \eta \in (a_1, b_1)$ the equality

$$\int_{a_2}^{b_2} \overline{\widetilde{H}(\xi, x)}\,\widetilde{H}_*(\eta, x)\,d\sigma_2(x) = \int_{a_1}^{b_1} e_\xi(x)e_\eta(x)\,d\sigma_1(x),$$

holds; moreover, the adjoint kernel $\widetilde{H}(\xi, x)$ is complete in H_1 (we call such kernels **B-kernels** (^{2})). It is connected with $\widetilde{K}_*(\xi, x)$ by the relation

$$\int_{a_1}^{b_1} \widetilde{K}_*(\eta, x)e_\xi(x)\,d\sigma_1(x) = \int_{a_2}^{b_2} \overline{\widetilde{H}(\xi, x)}e_\eta(x)\,d\sigma_2(x) \quad (\xi \in (a_1, b_1),\ \eta \in (a_2, b_2)).$$

Further, to every function $f(x) \in H_1$ there corresponds a certain function $g(x) \in H_2$ such that, for all $\xi \in (a_1, b_1)$,

$$\int_{a_1}^{b_1} f(x) e_\xi(x)\, d\sigma_1(x) = \int_{a_2}^{b_2} \overline{\widetilde{H}(\xi, x)}\, g(x)\, d\sigma_2(x), \tag{11}$$

and, for all $\xi \in (a_2, b_2)$,

$$\int_{a_2}^{b_2} g(x) e_\xi(x)\, d\sigma_2(x) = \int_{a_1}^{b_1} \overline{\widetilde{K}(\xi, x)}\, f(x)\, d\sigma_1(x). \tag{12}$$

Finally, the correspondence between the elements $f(x) \in H_1$ and $g(x) \in H_2$ (or $g^*(x) \in H_2$), realized by formulas (11) and (12) (respectively (9) and (10)), is quasi-isometric[2], and there exist positive constants M, m, K, k such that the inequalities

$$m\|f\|_{\sigma_1} \le \|g\|_{\sigma_2} \le M\|f\|_{\sigma_1}, \qquad k\|f\|_{\sigma_1} \le \|g^*\|_{\sigma_2} \le K\|f\|_{\sigma_1}.$$

Theorem 4. Let $\widetilde{K}(\xi, x)$ be a Riesz kernel admitting the representation

$$\widetilde{K}(\xi, x) = \int_{a_2}^{b_2} \varphi(t, x) e_\xi(t)\, d\sigma_2(t) \qquad (x \in (a_1, b_1),\ \xi \in (a_2, b_2)),$$

and let $\widetilde{K}_1(\xi, x)$ be some complete kernel in H_1 of the form

$$\widetilde{K}_1(\xi, x) = \int_{a_2}^{b_2} \psi(t, x) e_\xi(t)\, d\sigma_2(t) \qquad (x \in (a_1, b_1),\ \xi \in (a_2, b_2)).$$

If

$$\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} |\varphi(x, t) - \psi(x, t)|^2\, d\sigma_2(x) \right) d\sigma_1(t) < \infty \tag{13}$$

and, for any α_1 and α_2 lying inside (a_2, b_2), the manifold of functions

$$\int_{\alpha_1}^{\alpha_2} [\varphi(t, x) - \psi(t, x)] g(t)\, d\sigma_2(t), \qquad \|g\|_{\sigma_2} \le 1,$$

is compact in H_1, then $\widetilde{K}_1(\xi, x)$ is a Riesz kernel.

Institute of Mathematics and Mechanics
Academy of Sciences of the Armenian SSR

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Note: Figure translations are in progress. See original paper for figures.

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