A NEW DEFINITION OF UNIFORM SPACES

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Abstract

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A NEW DEFINITION OF UNIFORM SPACES

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At the basis of the proposed definition lies the concept of a system of sets with small ones, i.e., of such a system that contains arbitrarily small subsets. It is known that this concept is meaningful for uniform spaces. It is easy to prove that a uniformly continuous mapping takes a system with small ones into a system with small ones; conversely, if a mapping preserves all systems with small ones, then it is uniformly continuous. This means that specifying the set of all systems with small ones determines the uniform structure. The idea of such a definition of a uniform structure belongs to A. S. Schwartz.

Let us pass to the precise exposition of the matter. For what follows the following definition is convenient. Let A and B be systems of subsets of a space X. We shall say that the system A **majorizes** the system B ($A > B$), if for every $\beta \in B$ there exists an $\alpha \in A$ such that $\alpha \subset \beta$. In particular, $A \supset B$ implies $A > B$. By the symbol $\widetilde{A}$ we shall denote the union of all elements of the system.

Let a filter of neighborhoods $\mathfrak{B}$ define on E a uniform structure. A system γ of nonempty subsets of E will be called a **system with small ones** if for every $V \in \mathfrak{B}$ there is an $M \in \gamma$ such that $M^2 = M \times M \subset V$. The set of all systems with small ones will be denoted by Γ. If $\gamma \notin \Gamma$, then we shall say that γ is a **system without small ones**. In particular, the empty system Λ is a system without small ones.

The concept of a regular subsystem plays the main role in what follows: if γ is a system with small ones, then, throwing out from it some system γ' without small ones, we obtain the system $\gamma_1 = \gamma \setminus \gamma'$, which is called a **regular subsystem** of the system γ. We shall use the notation

$$\gamma_1 \subset \gamma.$$

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It is easy to see that a regular subsystem is also a system with small ones.

Let us note the obvious proposition:

1. *If* $\gamma \in \Gamma$, $V \in \mathfrak{B}$, γ_V *is the set of those* $M \in \gamma$ *for which* $M^2 \subset V$, *then*

$$\gamma_V \subset \gamma.$$

reg

We shall say that a system γ' is **confinal** to a system with small ones γ, if in each

$$\gamma_1 \subset \gamma$$

reg

there is contained at least one element γ'. Analogously, a system T of pairs of subsets is confinal to γ, if every

$$\gamma_1 \subset \gamma$$

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contains at least one pair $\tau \in T$. Directly from the definition of confinality there follows the proposition:

2. *If the system γ' is confinal to $\gamma \in \Gamma$, then $\gamma' \in \Gamma$.*

Indeed, if $\gamma' \notin \Gamma$, then $\gamma_1 = \gamma \setminus \gamma'$ is a regular subsystem of the system γ, not containing elements of γ'.

For an axiomatization of the concept of a system with small ones we single out the following properties:

$$\Gamma 1. \quad \text{If } \gamma_1 \text{ majorizes } \gamma \in \Gamma, \text{ then } \gamma_1 \in \Gamma.$$

$$\Gamma 2. \quad \text{If } \gamma \text{ consists of one singleton set, then } \gamma \in \Gamma.$$

\$ 2’ \$ (separation axiom). If $\gamma \in \Gamma$ consists of a single set M, then M is a singleton.

\$ \$3. From $\gamma_1 \cup \gamma_2 \in \Gamma$ it follows that one of the summands is a system with small sets.

\$ \$3 may be replaced by the equivalent proposition:

\$ \$3a. If $\gamma_1 \subset_{\text{reg}} \gamma$, then $\gamma_1 \in \Gamma$.

\$ \$4. If $\gamma \in \Gamma$, then γ is majorized by some system γ' of two-point sets, $\gamma' \in \Gamma$.

In what follows we shall agree to call a pair of intersecting subsets simply a “pair.”

\$ \$5. If a system T of pairs is cofinal with the system $\gamma \in \Gamma$, then the system γ_T, obtained from T by replacing each pair by the sum of its elements, is a system with small sets.

Let us note that here one may restrict oneself to the case when the pairs of the system T are formed by two-point sets. In this case the connection of \$ \$5 with the triangle axiom becomes still more transparent.

Let us prove \$ \$5. We must verify that for every $V \in \mathfrak{B}$ there exists an element $M \in \gamma_T$ whose square is contained in V. Take such a $W \in \mathfrak{B}$ that

$$W^2 \subset V.$$

By virtue of 1, the set γ_W of those elements γ whose squares are contained in W is a regular subsystem of the system γ. By hypothesis there exists a pair from Γ contained in γ_W. Let the elements of this pair be M_1 and M_2. Then $M_1^2, M_2^2 \subset W$. The set $M = M_1 \cup M_2$ is the element of γ_T that we need. Indeed, let $x, z \in M$, and let $y \in M_1 \cap M_2$. Since $(x, y), (y, z) \in W$, it follows that $(x, z) \in W^2 \subset V$. Thus \$ \$5 is proved.

Definitions. If a set Γ of nonempty systems γ of nonempty subsets of E is given, and if axioms \$ \$1–5 are satisfied, then we shall say that a uniform structure is defined on E. If \$ 2' \$ is satisfied, the structure is called separated. A system $\gamma \in \Gamma$ is called a system with small sets.

Let us prove the equivalence of our definition with Weil' s definition. Every structure $\mathfrak{B}$ in the sense of Weil generates, in the manner described above, a structure Γ in the new sense. We must prove that every structure Γ is generated by one and only one structure $\mathfrak{B}$. The uniqueness of $\mathfrak{B}$ follows from the fact that every mapping preserving Γ is uniformly continuous and, consequently, preserves $\mathfrak{B}$. Let us prove the existence of $\mathfrak{B}$. First note one obvious property of regular subsystems, following from \$ \$3.

3. If $\gamma_1, \gamma_2 \subset_{\text{reg}} \gamma$, then $\gamma_1 \cap \gamma_2 \subset_{\text{reg}} \gamma$.

This property proves that the set of all regular subsystems of a system $\gamma \in \Gamma$ forms a directed set with the natural order relation. Thus the definition of cofinality given above is justified.

Proposition 1 suggests the following way of specifying the set $\mathfrak{B}$ of subsets of $E \times E$: $V \in \mathfrak{B}$ if, for every $\gamma \in \Gamma$, there exists such a $\gamma_1 \subset_{\text{reg}} \gamma$ that $M \in \gamma_1$ implies $M^2 \subset V$. We shall prove that $\mathfrak{B}$ is a filter of neighborhoods (for the axioms, see (1)). The filter axioms (F_1) and (F_2) easily follow from the definition of $\mathfrak{B}$ and 3.

Let us verify the axioms of the neighborhood filter:

(U_I) Every $V \in \mathfrak{B}$ contains the diagonal.

(U_{Ia}) The intersection of all neighborhoods is the diagonal.

$$(U_{\mathrm{II}})\quad \text{From } V \in \mathfrak{B} \text{ it follows that } V^{-1} \in \mathfrak{B}.$$

$$(U_{\mathrm{III}})\quad \text{For every } V \in \mathfrak{B} \text{ there exists such a } W \in \mathfrak{B} \text{ that } W^2 \subset V.$$

(U_I) follows from \$ \$2, (U_{Ia}) follows from \$ 2—2' \$, (U_{II}) follows from the symmetry of the square. The verification of (U_{III}) is more complicated.

To each system γ of two-point subsets of E there naturally corresponds a system $\mathfrak{M}_\gamma$ of one-point subsets of $E \times E$: the set $\{(x, y)\}$ is an element of $\mathfrak{M}_\gamma$ if $\{x, y\} \in \gamma$. Similarly, to each system $\mathfrak{M}$ of one-point subsets of $E \times E$ there corresponds a system $\gamma_{\mathfrak{M}}$ of two-point or one-point subsets of E: $\{x, y\} \in \gamma_{\mathfrak{M}}$ $(x \neq y)$, if $\{(x, y)\} \in \mathfrak{M}$; $\{x\} \in \gamma_{\mathfrak{M}}$, if $\{(x, x)\} \in \mathfrak{M}$.

4. *If a system γ of two-point sets is a system without small sets, then*

$$V = E \times E \setminus \widetilde{\mathfrak{M}}_\gamma \in \mathfrak{B}.$$

Suppose the contrary. Then there exists $\gamma' \in \Gamma$ such that for no $\gamma_1 \supset \gamma'$ and $N \in \gamma_1$ does it follow that $N^2 \subset V$. In other words, in every $\gamma_1 \subset \gamma'$ there are elements whose squares intersect $\widetilde{\mathfrak{M}}_\gamma$. Select from γ' the set γ_0 of the indicated elements. Since γ_0 is cofinal in γ', $\gamma_0 \in \Gamma$, and since the square of each element of γ_0 contains an element of $\mathfrak{M}_\gamma$, each element of γ_0 contains an element of γ, i.e. γ majorizes γ_0. But then it follows from $\Gamma 1$ that γ is a system with small sets, which contradicts the hypothesis.

Proposition 4 can be formulated differently:

4$'$. *If a system γ of two-point sets is a system without small sets, then $\mathfrak{M}_\gamma$ does not majorize $\mathfrak{B}$.*

Hence it follows:

5. *If a system $\mathfrak{M}$ of one-point sets majorizes $\mathfrak{B}$, then $\gamma_{\mathfrak{M}}$ is a system with small sets.*

We now prove (U_{III}). Suppose that for some $V \in \mathfrak{B}$ there does not exist such a $W \in \mathfrak{B}$ that $W^2 \subset V$. Then every $W \in \mathfrak{B}$ contains points (x_W, y_W) and (y_W, z_W) such that $(x_W, z_W) \notin V$. Let

$$\begin{gathered} A_W = \{x_W, y_W\}, \quad B_W = \{y_W, z_W\}, \quad C_W = \{x_W, z_W\}, \\ \gamma_A = \{A_W\}, \quad \gamma_B = \{B_W\}, \quad \gamma_C = \{C_W\} \quad (W \in \mathfrak{B}). \end{gathered}$$

From the definition of $\mathfrak{B}$ it follows that $\gamma_C \in \Gamma$. By virtue of 5, $\gamma_A, \gamma_B \in \Gamma$. Denote by τ_W the pair $\{A_W, B_W\}$. We verify that the system $T = \{\tau_W\}_{W \in \mathfrak{B}}$ is cofinal with the system $\gamma = \gamma_A \cup \gamma_B$. Let $\gamma_1 \subset \gamma$, $\gamma_2 = \gamma \setminus \gamma_1$. If the system γ_2 contains at least one element from each pair τ_W, then the system $\mathfrak{M}_{\gamma_2}$ majorizes

$\mathfrak{B}$. By virtue of 5, $\gamma_2 \in \Gamma$, and since γ_2 is a system without small sets, this proves the existence of a pair τ_W contained in γ_1. On the basis of $\Gamma 5$, the system $\gamma_0 = \{d_W\}$ $(W \in \mathfrak{B})$, where $d_W = \{x_W, y_W, z_W\}$, is a system with small sets, and since γ_C majorizes γ_0, a contradiction with $\Gamma 1$ is obtained. Finally, we prove that the set Γ' of systems with small sets generated by the structure $\mathfrak{B}$ coincides with Γ. From the definition of $\mathfrak{B}$ it follows at once that $\Gamma \subset \Gamma'$. Let $\gamma \notin \Gamma$, and let $\gamma' \notin \Gamma$ be a system of two-point sets majorizing γ $(\Gamma 4)$. By virtue of 4, the system $\mathfrak{M}_{\gamma'}$ does not majorize $\mathfrak{B}$, and this means that $\gamma' \notin \Gamma'$. All the more, $\gamma \notin \Gamma'$. The proof of the equivalence is complete.

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REFERENCES

1. N. Bourbaki, *General Topology, Basic Structures*, 1958.

Note: Figure translations are in progress. See original paper for figures.

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