CONSTRUCTION OF ASYMPTOTIC EXPANSIONS AT WEAK INTERACTION FROM THE FORMAL THERMODYNAMIC PERTURBATION THEORY IN A MODIFIED FORMULATION OF THE PROBLEM OF A NONIDEAL BOSE-EINSTEIN SYSTEM

`![Fig. 1](#)`

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

MATHEMATICAL PHYSICS

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CONSTRUCTION OF ASYMPTOTIC EXPANSIONS AT WEAK INTERACTION FROM THE FORMAL THERMODYNAMIC PERTURBATION THEORY IN A MODIFIED FORMULATION OF THE PROBLEM OF A NONIDEAL BOSE–EINSTEIN SYSTEM

(Presented by Academician N. N. Bogolyubov, June 10, 1960)

A natural question arises: to what approximation does the summation of diagrams of a special type, appearing in the statistical variational principle (see (1)), correspond? It seems to us that the chief merit of this kind of summation, which restructures formal perturbation theory, consists in the fact that with its aid one can construct asymptotic expansions, at weak interaction, of the quantities of interest to us; i.e., it provides us with a regular method for constructing expansions of this type.

Fig. 1

In the first approximation, corresponding to taking into account the two first-order diagrams shown in Fig. 1, we must solve simultaneously the following system of nonlinear integral equations:

$$A(p) = E(p) + n_0 v(p) + \frac{1}{2V}\sum_{p'}(v(p-p') - v(p'))\left(\frac{A(p')}{\varepsilon(p')}\operatorname{cth}\frac{\beta\varepsilon(p')}{2} - 1\right) +$$

$$+\frac{1}{2V}\sum_{p'} v(p')\frac{B(p')}{\varepsilon(p')}\operatorname{cth}\frac{\beta\varepsilon(p')}{2} + \dots, \tag{1}$$

$$B(p) = n_0 v(p) - \frac{1}{2V}\sum_{p'} v(p-p')\frac{B(p')}{\varepsilon(p')}\operatorname{cth}\frac{\beta\varepsilon(p')}{2}, \tag{2}$$

$$\varepsilon^2(p) = A^2(p) - B^2(p). \tag{3}$$

Knowing its solution, we can calculate

$$\mu = nv(0) + \frac{1}{2V}\sum_{p'} v(p')\left(\frac{A(p')}{\varepsilon(p')}\operatorname{cth}\frac{\beta\varepsilon(p')}{2} - 1\right) -$$

$$-\frac{1}{2V}\sum_{p'} v(p')\frac{B(p')}{\varepsilon(p')}\operatorname{cth}\frac{\beta\varepsilon(p')}{2} + \ldots, \tag{4}$$

$$n = n_0 + \frac{1}{2V}\sum_{p'}\left(\frac{A(p')}{\varepsilon(p')}\operatorname{cth}\frac{\beta\varepsilon(p')}{2} - 1\right) + \ldots \tag{5}$$

Solving asymptotically, for small v and fixed n_0, the nonlinear equations (1), (2), (3), we obtain, in the case $\theta = 0$,

$$\mu = nv(0) - \frac{n_0}{2V}\sum_p \frac{v^2(p)}{E(p)} + \frac{4}{3\pi^2}\frac{m^{3/2}n_0^{3/2}v^{5/2}(0)}{\hbar^3} +$$

$$+\frac{n_0}{4V^2}\sum_{p,p'}\frac{v(p)v(p-p')v(p')}{E(p)E(p')} + \frac{3n_0^2}{4V}\sum_p\frac{v^3(p)-v^3(0)}{E^2(p)} -$$

$$-\frac{28}{15\pi^2}\frac{8m^{5/2}n_0^{5/2}\nu^{5/2}(0)\nu'(0)}{\hbar^3} - \frac{2}{\pi^2}\frac{m^{3/2}n_0^{3/2}\nu^{3/2}(0)}{\hbar^3}\frac{1}{2V}\sum_p\frac{\nu^2(p)}{E(p)} +$$

$$+\frac{n_0^2}{4V^2}\sum_{p,p'}\frac{\nu^2(p)(\nu^2(p')-\nu(p-p')\nu(p'))}{E^2(p)E(p')} + \frac{n_0^2}{2V^2}\sum_{p,p'}\frac{\nu^2(0)\nu^2(p')-\nu^2(p)\nu(p-p')\nu(p')}{E(p')E^2(p)} -$$

$$-\frac{n_0}{8V^3}\sum_{p,p',p''}\frac{\nu(p)\nu(p-p')\nu(p'-p'')\nu(p'')}{E(p)E(p')E(p'')} + \frac{1}{\pi^4}\frac{m^3n_0^2\nu^4(0)}{\hbar^6} +$$

$$+\frac{1}{2V}\sum_p\left(-\frac{5n_0^3\nu^4(p)}{2E^3(p)} + \frac{5n_0^3\nu^4(0)}{2E^3(p)} + 20m\nu'(0)\frac{n_0^3\nu^3(0)}{E^2(p)}\right) + \ldots; \tag{6}$$

$$n = n_0 + \frac{1}{3\pi^2}\frac{m^{3/2}n_0^{3/2}\nu^{3/2}(0)}{\hbar^3} + \frac{n_0^2}{2V}\sum_p\frac{\nu^2(p)-\nu^2(0)}{2E^2(p)} + \ldots \tag{7}$$

The terms ν and ν^2 in (6) are obtained in ordinary perturbation theory. The terms $\nu^{5/2}$ are in agreement with those obtained by Lee, Huang, and Yang ([2]). The terms ν^3 coincide with those obtained by Girardeau ([3]). The terms $\nu^{7/2}$ and

ν^4 are not guaranteed in the first approximation, since there are contributions of this type from the second approximation.

In the case $\theta \neq 0$, asymptotically for small ν and fixed n_0 we obtain

$$\mu = n\nu(0) + \frac{1}{2V}\sum_p \nu(p)\left(\operatorname{cth}\frac{\beta E(p)}{2} - 1\right) - \frac{m^{1/2}n_0^{1/2}\nu^{3/2}(0)\theta}{\pi\hbar^3} -$$

$$-\frac{1}{2V}\sum_p \nu(p)\frac{\beta}{2}\frac{1}{\operatorname{sh}^2\frac{\beta E(p)}{2}}\left\{n_0\nu(p) + \frac{1}{2V}\sum_{p'}(\nu(p-p') - \nu(p'))\left(\operatorname{cth}\frac{\beta E(p')}{2} - 1\right)\right\} +$$

$$+\frac{1}{2V}\sum_p \frac{4n_0\nu^2(0)}{\beta E^2(p)} - \frac{1}{2V}\sum_p \frac{n_0\nu^2(p)}{E(p)}\operatorname{cth}\frac{\beta E(p)}{2} + \dots; \qquad (8)$$

$$n = n_0 - \frac{1}{2\pi\hbar^3}m^{3/2}n^{1/2}\nu^{1/2}(0)\theta - \frac{1}{(2\pi\hbar^3)^{1/2}}m^{3/4}n_0^{1/4}\nu^{3/4}(0)\theta^{3/2} +$$

$$+\frac{1}{8\pi^2\hbar^6}m^3\nu(0)\theta^2 + \frac{1}{2V}\sum_p\left(\operatorname{cth}\frac{\beta E(p)}{2} - 1\right) -$$

$$-\frac{1}{2V}\sum_p \frac{\beta}{2}\frac{1}{\operatorname{sh}^2\frac{\beta E(p)}{2}}\left\{n_0\nu(p) + \frac{1}{2V}\sum_{p'}(\nu(p-p') - \nu(p'))\left(\operatorname{cth}\frac{\beta E(p')}{2} - 1\right)\right\} + \qquad (9)$$

$$+\frac{1}{2V}\sum_p \frac{2}{\beta}\frac{n_0\nu(0)}{E^2(p)} + \dots$$

The terms ν^2 are not guaranteed, since there are contributions of this type from the second approximation.

Let us note that, in the modified formulation,

$$\frac{F}{V} = \left.\frac{F}{V}\right|_{n_0=0} - \int_0^{n_0} dn_0\, n\frac{d\mu}{dn_0}. \qquad (10)$$

In the case $\theta \neq 0$, using (10), asymptotically for small ν and fixed n_0 we obtain

$$\frac{F}{V} = \left.\frac{F}{V}\right|_{n_0=0} + \frac{1}{3\pi\hbar^3}m^{3/2}n_0^{1/2}\nu^{3/2}(0)\theta + \frac{1}{\pi\hbar^3}m^{3/2}n_0^{1/2}\nu^{3/2}(0)\theta\,\frac{1}{2V}\sum_p\left(\operatorname{cth}\frac{\beta E(p)}{2} - 1\right) +$$

$$+\mathrm{F.P.}\,\frac{1}{2V}\sum_p n_0^2 v^2(p)\,\frac{\beta}{4}\,\frac{1}{\operatorname{sh}^2\frac{\beta E(p)}{2}} + \mathrm{F.P.}\,\frac{1}{2V}\sum_p \frac{n_0^2 v^2(p)}{2E(p)}\,\operatorname{cth}\frac{\beta E(p)}{2} +$$

Fig. 2

Figure 2: Fig. 2

$$+\frac{1}{2V}\sum_p\left(\operatorname{cth}\frac{\beta E(p)}{2}-1\right)\mathrm{F.P.}\,\frac{1}{2V}\sum_p n_0 v^2(p)\,\frac{\beta}{2}\,\frac{1}{\operatorname{sh}^2\frac{\beta E(p)}{2}}+$$

$$+\frac{1}{2V}\sum_p\left(\operatorname{cth}\frac{\beta E(p)}{2}-1\right)\mathrm{F.P.}\,\frac{1}{2V}\sum_p \frac{n_0 v^2(p)}{E(p)}\operatorname{cth}\frac{\beta E(p)}{2}-$$

$$-\frac{1}{4\pi^2\hbar^6}\,m^3 n_0 v^2(0)\,\theta^2+\cdots, \tag{11}$$

where F. P. denotes the finite part of the integral divergent at small p*.

We pass to the second approximation. The starting formula for its calculation is

$$-\beta\Delta F=\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2\,\langle\Omega_{\text{int}}(\tau_1)\Omega_{\text{int}}(\tau_2)\rangle_{\text{conn}}, \tag{12}$$

Fig. 2

where the subscript conn means that only connected diagrams are to be taken into account,

$$\Omega_{\text{int}}(\tau)=e^{\tau\Omega_0}\Omega_{\text{int}}e^{-\tau\Omega_0},\qquad \langle\cdots\rangle=\operatorname{Sp}e^{-\beta\Omega_0}\cdots\big/\operatorname{Sp}e^{-\beta\Omega_0}.$$

For the contribution from the most important diagram in second order, shown in Fig. 2 (the second-order diagram leading to logarithmic terms in the interaction), we have

$$\Delta F_G=-\frac{n_0}{6V}\sum_{\substack{p_1,p_2,p_3\\p_1+p_2+p_3=0}}(1-3T_1)\frac{(1+n_{p_1})(1+n_{p_2})(1+n_{p_3})-n_{p_1}n_{p_2}n_{p_3}}{\varepsilon(p_1)+\varepsilon(p_2)+\varepsilon(p_3)}\times$$

$$\times\left\{(v(p_1)+v(p_2))(u_{p_3}v_{p_1}v_{p_2}+v_{p_3}u_{p_1}u_{p_2})+\text{symm}\right\}^2, \tag{13}$$

where symm denotes two more terms obtained by the cyclic permutation $(1,2,3)$; T_1 denotes the following operation, referring to the momentum p_1: $\varepsilon(p_1)\to-\varepsilon(p_1)$ and $u_{p_1}\rightleftarrows v_{p_1}$.

A study of the asymptotics of (13) for small v and fixed n_0 leads to the following formulas. In the case of temperature $\theta = 0$,

$$\frac{\Delta F_G}{V} = -\frac{1}{3\pi^2}\frac{m^{3/2}n_0^{5/2}v^{3/2}(0)}{\hbar^3}\frac{1}{2V}\sum_p \frac{v^2(p)}{E(p)} + \frac{1}{16\pi^2\hbar^6}m^3 n_0^3 v^4(0)\left(\frac{4}{3}-\frac{\sqrt{3}}{\pi}\right)\ln v(0) + \cdots \tag{14}$$

The second term on the right-hand side exactly corresponds to that obtained by Wu (5), Hugenholtz and Pines (6), and Sawada (4). In the case of temperature $\theta \neq 0$,

$$\frac{\Delta F_G}{V} = \frac{n_0}{6\pi^2\hbar^6}m^3 v^2(0)\theta^2\left(1-\frac{\sqrt{3}}{\pi}\right)\ln\frac{n_0 v(0)}{\theta} + \cdots. \tag{15}$$

The results obtained can be generalized in two directions:

1. To carry out a "ladder" summation, i.e., to replace the interaction by that renormalized as a result of the "ladder" summation, and thereby connect our treatment with the so-called pair-collision approximation often discussed in the literature.
2. To carry out a "density" summation for the study of collective excitations, equivalent to the random-phase approximation well known in plasma theory.

For example, $\quad \mathrm{F.\,P.}\,\frac{1}{V}\sum_p \frac{f(p)}{E^2(p)} = \frac{1}{V}\sum_p \frac{f(p)-f(0)}{E^2(p)} \quad$ (see (3)).

It seems to us, however, that improvements of this kind do not eliminate the principal shortcoming of the expansions under consideration: namely, their inadequacy, revealed upon more detailed study, in the range of temperatures close to the critical one.

Note added in proof. At present we have succeeded in making a more accurate estimate of $\Delta F_G/V$ in the case $\theta = 0$, which made it possible to write out in the chemical potential (and also in the ground-state energy) all terms of order ν^4. These terms were partly investigated by Wu ([5]). In view of the cumbersome nature of the indicated terms, we do not present them here.

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Received
7 VI 1960

REFERENCES

[1] V. V. Tolmachev, DAN, **134**, No. 6 (1960).
[2] T. D. Lee, K. Huang, C. N. Yang, Phys. Rev., **106**, 1135 (1957).
[3] M. Girardeau, Phys. Rev., **115**, 1090 (1959).
[4] K. Sawada, Phys. Rev., **116**, 1344 (1959).
[5] T. T. Wu, Phys. Rev., **115**, 1390 (1959).
[6] N. M. Hugenholtz, D. Pines, Phys. Rev., **116**, 489 (1959).

Note: Figure translations are in progress. See original paper for figures.