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Abstract

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LINEAR APPROXIMATION OF THE VELOCITY IN THE PROBLEM OF ONE-DIMENSIONAL MOTION OF A PLASMA WITH FINITE CONDUCTIVITY

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The system of equations of magnetic gas dynamics for plane one-dimensional motion
$\{\mathbf{v} = (u, 0, 0);\ \mathbf{H} = (0, H, 0)\}$, in the case of finite constant conductivity, sufficiently large that the displacement current and the convection current may be neglected, can be written in the form:

$$\begin{gathered}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(u\rho) = 0, \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{h}{\rho}\frac{\partial h}{\partial x} = 0, \\
\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + kp\frac{\partial u}{\partial x} = (k-1)\chi\left(\frac{\partial h}{\partial x}\right)^2, \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = \chi\frac{\partial^2 h}{\partial x^2}
\end{gathered} \tag{1}$$

(here $h = H/\sqrt{4\pi}$, $k = c_p/c_v$, $\chi = c^2/4\pi\sigma$ is the magnetic viscosity).

We have neglected terms containing viscosity and thermal conductivity. Exact criteria indicating when such an idealization is possible are contained, for example, in works ([1,2]).

In view of the considerable difficulties associated with finding exact solutions of the equations of magnetic gas dynamics, it is of interest to construct approximate solutions in which the velocity is prescribed as a function of some definite form. Such an approach is useful, in particular, in studying the motion of a plasma under the action of a magnetic field prescribed at the plasma boundary (the so-called "magnetic piston").

The plane one-dimensional problem of a magnetic piston is encountered in the theoretical consideration of experiments with electromagnetic shock tubes ([3,4]).

Owing to finite conductivity, the accelerating action of the magnetic field is partially eliminated.

In the work of M. A. Leontovich and S. M. Osovets (^{5}) (for the cylindrical case), the first step in this direction was taken. The motion of the entire mass of gas as a whole was considered, and the question of the distribution of velocity over the cross section was not discussed. K. P. Stankovich (^{6}) considered the case in which the velocity depends only on time ($u = a(t)$). In the work of L. A. Artsimovich (^{7}), the motion of the plasma boundary and the magnetic field at the boundary were assumed to be prescribed ($H_0 = \text{const}$). The velocity was prescribed as a function of the form $u = a(t)r$.

Experiments show that the velocity of plasma motion in the region between the shock-wave front and the magnetic piston can be rather well approximated by a linear function of x, where the coefficients are arbitrary functions of time. The essential advantage of such an approximation over those listed above is that it gives us a sufficient number of arbitrary functions for satisfying the boundary conditions.

Thus, let

$$u(x,t) = a(t)x + b(t). \tag{2}$$

Put $h = \partial\varphi/\partial x$; then the equation for the magnetic field takes the form

$$\frac{\partial\varphi}{\partial t} + u\frac{\partial\varphi}{\partial x} = \chi\frac{\partial^2\varphi}{\partial x^2}. \tag{3}$$

We now carry out a change of variables, introducing the Lagrangian coordinate x_0 and an effective time scale τ

$$\tau = \int_0^t \exp\left(-2\int a\,dt\right)dt. \tag{4}$$

The relation between the Eulerian and Lagrangian coordinates is obtained by integrating the expression for the velocity (2):

$$x = \exp\left(\int a\,dt\right)\left[x_0 + \int_0^t b(t)\exp\left(-\int a\,dt\right)dt\right],$$

$$x_0 = x\exp\left(-\int a\,dt\right) - \int_0^t b(t)\exp\left(-\int a\,dt\right)dt. \tag{5}$$

The Jacobian of the transformation is

$$D(\tau, x_0)/D(t,x) = \exp\left(-3\int a\,dt\right) \neq 0.$$

For $t = 0$, $x = x_0$. Hence the additive constant

$$s(0) = \int a(t)\,dt\Big|_{t=0} = 0.$$

In Lagrangian coordinates the magnetic field turns out to be specified at the point $x_0 = 0$.

Equation (5) is now reduced to the classical form of the heat-conduction equation for a homogeneous rod:

$$\frac{\partial\varphi}{\partial\tau} = \chi\frac{\partial^2\varphi}{\partial x_0^2}. \tag{6}$$

In the coordinates $\{x_0, \tau\}$ the usual skin effect takes place.

The boundary and initial conditions for the magnetic field in the magnetic-piston problem have the form

$$h(X_\text{p}(t), t) = \mu(t); \qquad h(x, 0) = 0, \tag{7}$$

where

$$X_\text{p}(t) = \exp\left(\int a\,dt\right)\int_0^t b(t)\exp\left(-\int a\,dt\right)dt \tag{8}$$

is the Eulerian coordinate of the piston (the plasma boundary).

The conditions at the front in our case have the form (8)

$$\rho_1(u_1 - D) = \rho_2(u_2 - D); \tag{9}$$

$$\rho_1(u_1 - D)^2 + p_1 + \frac{1}{2}h_1^2 = \rho_2(u_2 - D)^2 + p_2 + \frac{1}{2}h_2^2; \tag{10}$$

$$\begin{aligned} &\frac{k}{k-1}\frac{p_1}{\rho_1} + \frac{(u_1 - D)^2}{2} + \left[\frac{h_1}{\rho_1} - \frac{\chi}{\rho_1(u_1 - D)}\left(\frac{\partial h}{\partial x}\right)_1\right]h_1 = \\ &= \frac{k}{k-1}\frac{p_2}{\rho_2} + \frac{(u_2 - D)^2}{2} + \left[\frac{h_2}{\rho_2} - \frac{\chi}{\rho_2(u_2 - D)}\left(\frac{\partial h}{\partial x}\right)_2\right]h_2; \end{aligned} \tag{11}$$

$$(u_1 - D)h_1 - \chi \left(\frac{\partial h}{\partial x}\right)_1 = (u_2 - D)h_2 - \chi \left(\frac{\partial h}{\partial x}\right)_2. \tag{12}$$

Here the quantities in front of the wave front are denoted by the subscript 1, those behind the front by the subscript 2; $D = D(t) = dX_\phi(t)/dt$ is the velocity of the shock-wave front.

If the conductivity is large, then one may assume that the magnetic field propagates in the semi-infinite space ($x_0 > 0$). Indeed, the equation of the magnetic field has the form (6) for the region $0 \leqslant x_0 \leqslant X_\phi$, where $X_\phi = X_\phi(t)$ is the coordinate of the shock-wave front, while for the region $x_0 \geqslant X_\phi$ the equation has the form

$$\frac{\partial \varphi}{\partial t} = \chi \frac{\partial^2 \varphi}{\partial x_0^2}. \tag{13}$$

For small t the function $A(t) = \exp\left(\int a\, dt\right) \simeq 1$, and the effective time scale τ almost coincides with the scale t. Consequently, in the whole half-space $x_0 > 0$ the form of the equation is the same. As t increases, the field at the front rapidly decreases, and the distribution of the magnetic field in the region between the piston and the shock-wave front is almost independent of the propagation of the magnetic field in the region ahead of the jump front. If one takes into account that the magnetic field somewhat accelerates the gas ahead of the front, the equations for the regions behind the front and ahead of the front will be even closer in form. Thus, in Lagrangian coordinates the magnetic field may be regarded as continuous. However, in Eulerian coordinates a jump in the magnetic field and in its derivative will be observed at the front, since the function $x_0(x, t)$ has a kink at the shock-wave front.

Thus, with sufficient accuracy, the magnetic field can be expressed in the form of the derivative of the Duhamel integral:

$$h(x_0, t) = \frac{x_0}{2\sqrt{\pi\chi}\, A} \int_0^t \frac{\mu(q)}{A(q)\left(\int_q^t A^{-2}\, dq\right)^{3/2}} \exp\left(-\frac{x_0^2}{4\chi \int_q^t A^{-2}\, dq}\right) dq. \tag{14}$$

Here and below it is convenient to regard $A(t)$ and $X_\text{p}(t)$ as the new approximating functions. The functions $a(t)$ and $b(t)$ are easily expressed through them:

$$a(t) = \frac{1}{A}\frac{dA}{dt}, \qquad b(t) = A \frac{d}{dt}\left(\frac{X_\text{p}}{A}\right). \tag{15}$$

Having determined $h(x_0, t)$, it is not difficult to find the pressure and density by using the continuity equation and either the equation of motion or the energy equation. The remaining equation, in the general case, is not satisfied exactly. However, it is possible to choose the approximating functions $a(t)$ and $b(t)$ in such a way that this equation is satisfied approximately in the region of interest to us, between the magnetic piston and the shock-wave front.

Let the initial conditions be constant:

$$\rho(x,0) = \rho_0; \qquad p(x,0) = p_0. \tag{16}$$

Integrating the continuity equation, we obtain:

$$\rho(x_0,t) = \rho_0/A. \tag{17}$$

The neglect of viscosity, thermal conductivity, and radiation affects the energy equation more strongly than the equation of motion. Therefore one should satisfy the equation of motion exactly, and the energy equation approximately, by subjecting the functions $A(t)$ and $X_\mathrm{p}(t)$ to the condition of energy balance at the shock-wave front.

At the piston, in essence, there is a contact discontinuity; consequently, the total pressure $p + \frac{1}{2}h^2$ must be continuous. Taking into account

this, from the equation of motion we obtain

$$p(x_0,t) = -\frac{1}{2}h^2 + \frac{1}{2}\mu^2(t) - \frac{1}{2}\rho_0 A'' x_0^2 - \rho_0 X_\mathrm{p}'' x_0. \tag{18}$$

Here primes denote differentiation with respect to t. We have three arbitrary functions: $A(t)$, $X_\phi(t)$, $X_\mathrm{p}(t)$, by means of which we can satisfy the discontinuity conditions (9), (10), (11).

The condition of mass balance at the shock-wave front gives an equation for $X_\phi(t)$, whence it is easily found that

$$X_\phi(t) = \frac{X_\mathrm{p}}{1-A}. \tag{19}$$

Let us now consider the momentum-balance equation (10). We note that in our case the right-hand side of the equation does not depend on h_2. Consequently, the left-hand side also must not depend on the seepage of the magnetic field through the front. It is therefore possible, without introducing any new error, to take $h_1 = 0$, and then $u_1 = 0$; $p_1 = p_0$; $\rho_1 = \rho_0$ (as would be the case of infinite conductivity ahead of the front). We obtain:

$$p_0 + \rho_0 \frac{\left[(1-A)X_{\mathrm{p}}' + A'X_{\mathrm{p}}\right]^2}{(1-A)^3} + \frac{\rho_0}{2}\frac{A''X_{\mathrm{p}}^2}{(1-A)^2} + \rho_0\frac{X_{\mathrm{p}}''X_{\mathrm{p}}}{1-A} = \frac{\mu^2(t)}{2}. \tag{20}$$

Using the momentum equation and neglecting the velocity of the medium ahead of the front, one can write the energy-balance equation in the form

$$\frac{p_0}{A} + \frac{k-1}{2k}\frac{\rho_0}{A}\frac{(1+A)\left[(1-A)X_{\mathrm{p}}' + A'X_{\mathrm{p}}\right]^2}{(1-A)^3} + \frac{\rho_0}{2}\frac{A''X_{\mathrm{p}}^2}{(1-A)^2} + \frac{\rho_0 X_{\mathrm{p}}''X_{\mathrm{p}}}{1-A} =$$

$$= \frac{\mu^2(t)}{2} + \left\{\left[Ah_2 - \frac{\varkappa A_2}{X_{\mathrm{p}}'}\left(\frac{\partial h}{\partial x}\right)_2\right](1-A) + \frac{(A^3+1)h_2}{2}\right\}\frac{h_2}{\rho_0}. \tag{21}$$

The system of equations (20), (21) determines the approximating functions $a(t)$, $b(t)$. The influence of the magnetic field behind the front on the motion of the front is substantial only for very small t. Investigating later stages of the process, one may discard the terms containing h. The equations then simplify, and, combining (20) and (21), it is easy to obtain an expression for the coordinate of the front

$$X_\phi(t) = \frac{X_{\mathrm{p}}}{1-A} = \sqrt{2c_0}\int_0^t \frac{dt}{\sqrt{A(k+1)-(k-1)}}, \tag{22}$$

where $c_0 = \sqrt{kp_0/\rho_0}$ is the speed of sound in the undisturbed medium. Hence it is seen that $1 \geq A \geq (k-1)/(k+1)$. This estimate remains valid for small t, since then $A \approx 1$.

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Note: Figure translations are in progress. See original paper for figures.