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Abstract

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MATHEMATICS

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A SYSTEM OF AXIOMS FOR SEMIGROUPS EMBEDDABLE IN GENERALIZED GROUPS

(Presented by Academician A. I. Mal' cev on 30 V 1960)

We use the notation of mathematical logic: $\wedge$ is the symbol of conjunction; $\rightarrow$ is the symbol of implication; $\vee$ is the existential quantifier; E is the symbol of abstraction; $\bigwedge$ is the universal quantifier. The conjunction of a finite set of propositions $B_k \wedge B_{k+1} \wedge \cdots \wedge B_{k+m}$ is denoted, for short, by $\bigwedge_{i=k}^{k+m} B_i$, where $\bigwedge_{i=k}^{k} B_i$ is B_k.

A generalized group is a semigroup such that for each element g there exists an element g^{-1} such that $gg^{-1}g = g \wedge g^{-1}gg^{-1} = g^{-1}$, and the idempotents of the semigroup commute pairwise ([1]). It is known that for each g in a generalized group there exists exactly one element g^{-1}, called the generalized inverse of g.

Between the elements of a generalized group G, one can introduce, in any of the following three ways ([1]), an order relation ω, called **canonical**:

$$\omega = E_{(g_1,g_2)}(g_1 g_1^{-1} g_2 = g_1) = E_{(g_1,g_2)}(g_1 g_2^{-1} g_1 = g_1) = E_{(g_1,g_2)}(g_2 g_1^{-1} g_1 = g_1).$$

A semigroup is called embeddable in a generalized group if this semigroup can be mapped isomorphically into some generalized group. It is not difficult to show that a semigroup is embeddable in a generalized group if and only if this semigroup is isomorphic to some semigroup of partial one-to-one transformations of some set. Therefore the system of axioms for the class of semigroups embeddable in generalized groups, which we shall seek, will also be a system of axioms for the class of all semigroups of partial one-to-one transformations of sets. Moreover, the problem of embedding a semigroup in a generalized group is akin to the problem of embedding a semigroup in a group, solved in the well-known papers of A. I. Mal' cev ([2,3]). Let us note that a group is a special case of a generalized group.

It suffices to consider only semigroups containing an identity. Indeed, if a semigroup D does not contain an identity, then, adjoining to it an element e and defining $ee = e,\ eg = ge = g$ for all $g \in D$, we obtain a semigroup D^* with identity e. It is easy to show that the semigroups D and D^* are simultaneously embeddable or not embeddable in a generalized group.

In what follows, only semigroups containing an identity are considered. By the letters u, v, x, y, z, with indices or without them, we shall denote individual variables over the set D of elements of the semigroup under consideration.

A quasi-order relation [(4)] ξ between elements of a semigroup D is called **stable** if it satisfies the condition*

$$xv \in \xi\langle z\rangle \wedge uv \in \xi\langle z\rangle \wedge uy \in \xi\langle z\rangle \to xy \in \xi\langle z\rangle.$$

For example, the universal binary relation $D \times D$ will, as is easy to see, be a stable quasi-order relation of the semigroup D.

The intersection of all stable quasi-order relations between elements of the semigroup D will obviously be a stable quasi-order relation. We shall call it the **strong quasi-order relation** of the semigroup D and denote it by $\hat{\xi}$.

It turns out that the strong quasi-order relation of a generalized group coincides with its canonical order relation.

Theorem 1. *In order that a semigroup be embeddable in a generalized group, it is necessary and sufficient that the strong quasi-order relation of this semigroup be an order relation.*

The embeddability condition given by Theorem 1 can be written in the form of the formula

$$\hat{\xi} \cap \hat{\xi}^{-1} \subset \Delta_D, \tag{1}$$

where Δ_D, as usual, denotes the identity binary relation between elements of the semigroup D.

Formula (1) may be regarded as a nonelementary condition characterizing the class of semigroups embeddable in generalized groups. Our task is to replace the nonelementary condition (1) by an equivalent system of elementary conditions.

Introduce binary relations $\xi_0, \xi_1, \ldots$ between elements of the semigroup D, defining

$$\xi_0 = \Delta_D,$$

$$\begin{aligned} \xi_m = \mathrm{E}_{(z_1,z_2)u,v,x,y}(\vee \ \ z_2 = xy \wedge xv \in \xi_{m-1}\langle z_1\rangle \wedge \\ uv \in \xi_{m-1}\langle z_1\rangle \wedge \\ uy \in \xi_{m-1}\langle z_1\rangle). \end{aligned} \tag{2}$$

It turns out that the formulas

$$\xi_{n_2} \circ \xi_{n_1} \subset \xi_{n_1+n_2}, \qquad \xi_m \subset \xi_{m+1}. \tag{3}$$

hold. But the most important is the formula

$$\hat{\xi} = \bigcup_{m=1}^{\infty} \xi_m. \tag{4}$$

Using this formula, one can prove, for example, that the strong quasi-order relation of a semigroup is stable, i.e. that

$$(x_1, x_2) \in \hat{\xi} \wedge (x_3, x_4) \in \hat{\xi} \to (x_1 x_3,\ x_2 x_4) \in \hat{\xi}.$$

It is easy to see that condition (1) is equivalent to the condition

$$\xi_n \cap \xi_n^{-1} \subset \Delta_D \quad \text{for all } n. \tag{5}$$

Let us find explicit expressions for the binary relations ξ_n, which were defined by the recurrence formulas (2).

(_\|\\|\\)

* We assume that conjunction binds statements more strongly than implication. Accordingly, parentheses in statements of the form $(A \wedge B) \to C$ will be omitted.

Let $u_j^i,\ v_j^i,\ x_j^i,\ y_j^i$ be individual variables over the set D of elements of the semigroup under consideration, where the index i (which should not be confused with an exponent) takes natural values from 1 to n (everywhere below, unless otherwise stated, n is a fixed natural number), and j takes positive or negative integer values.

We shall agree to understand by x_j^{n+1} the variable x_{-j}^1, if j is positive, and x_j^1, if j is negative. In other words, $x_j^{n+1} = x^1_{-\operatorname{sign} j}$. Analogously, put $y_j^{n+1} = y^1_{-\operatorname{sign} j}$. The convention introduced will considerably simplify the notation of formulas.

By induction on n it can be shown that

$$\zeta_n = \mathop{\mathrm{E}}_{(z_1,z_2)} \left(\bigvee_{u_j^i, v_j^i, x_j^i, y_j^i} z_2 = x_1^1 y_1^1 \wedge \bigwedge_{i=1}^{n} \bigwedge_{k=1}^{3^i-1} \begin{pmatrix} x_k^i v_k^i = x_{3k-2}^{i+1} y_{3k-2}^{i+1} \wedge \\ u_k^i v_k^i = x_{3k-1}^{i+1} y_{3k-1}^{i+1} \wedge \\ u_k^i y_k^i = x_{3k}^{i+1} y_{3k}^{i+1} \end{pmatrix} \wedge x_{-1}^1 y_{-1}^1 = z_1 \right).$$

This formula, together with formula (4), makes it possible to define the strong quasiorder relation of a semigroup in an elementary way. The same formula makes it possible to write relation (5) in elementary form. Relation (5) is equivalent to

$$\bigwedge_{i=1}^{n} \bigwedge_{k=1}^{3^i-1} \begin{pmatrix} x^i_k v^i_k = x^{i+1}_{3k-2} y^{i+1}_{3k-2} \wedge x^i_{-k} v^i_{-k} = x^{i+1}_{2-3k} y^{i+1}_{2-3k} \wedge \\ u^i_k v^i_k = x^{i+1}_{3k-1} y^{i+1}_{3k-1} \wedge u^i_{-k} v^i_{-k} = x^{i+1}_{1-3k} y^{i+1}_{1-3k} \wedge \\ u^i_k y^i_k = x^{i+1}_{3k} y^{i+1}_{3k} \wedge u^i_{-k} y^i_{-k} = x^{i+1}_{-3k} y^{i+1}_{-3k} \end{pmatrix} \to x^1_{-1} y^1_{-1} = x^1_1 y^1_1. \quad (6)$$

Here, when replacing ζ_n^{-1}, which enters into relation (5), by an elementary expression, individual variables with negative lower indices have been taken. Formula (6), for fixed n, will be denoted by A_n.

Now the following may be formulated.

Theorem 2. *The sequence of axioms $A_1, A_2, \ldots$ characterizes the class of semigroups embeddable in generalized groups. (6) is an axiom scheme.*

It should be noted that the axioms $\{A_n\}$ were derived under the assumption that the semigroups under consideration contain an identity. It can be shown that a semigroup D (not necessarily containing an identity) which satisfies the axioms $\{A_n\}$ is embeddable in a generalized group if and only if this semigroup is reductive, i.e., satisfies the condition

$$\left(\bigwedge_{g, g_0} g g_1 g_0 = g g_2 g_0 \right) \to g_1 = g_2.$$

Let us also note that reductivity is not a necessary condition for embeddability of a semigroup in a generalized group.

The question of independence of the constructed system of axioms naturally arises. From formula (3) it is easy to see that the axiom A_n entails all A_m for $m \leqslant n$. Hence it follows at once that any infinite subsequence of the sequence $\{A_n\}$ is itself a system of axioms for the class of semigroups embeddable in generalized groups.

Construct an associative calculus with alphabet u^i_j, v^i_j, x^i_j, y^i_j, where $|j| = 1, 2, \ldots, 3^i - 1$ and $i = 1, 2, \ldots, n$, and with the axiom A_n as the set of defining relations (5). This associative calculus generates a semigroup in which, as is not difficult to verify, the axiom A_n is not satisfied. By means of a fairly long chain of arguments one can show that the axiom A_{n-1} is satisfied in the constructed semigroup. Therefore the axiom A_n does not follow from the axiom A_{n-1} for any n, beginning with 2. Thus the system of axioms we have constructed is essentially infinite.

in the sense that it is not equivalent to any of its finite subsequences.

Let us summarize what has been said in two theorems:

Theorem 3. *Let A_m and A_n be two axioms and $m < n$. Then A_n implies A_m, but A_m does not imply A_n.*

Theorem 4. *No finite system of elementary axioms can characterize the class of semigroups embeddable in generalized groups.*

Suppose that the class of semigroups embeddable in generalized groups can be characterized by a finite system of axioms $B_1, B_2, \dots, B_k$. From the completeness of elementary theory it follows that the formula $B_1 \wedge B_2 \wedge \dots \wedge B_k$ can be derived from the formulas $\{A_n\}$. Since in this derivation only finitely many of the axioms $\{A_n\}$ can be used, the system $\{A_n\}$ is equivalent to one of its finite subsystems, which, as we have seen, is false. The contradiction obtained completes the proof of the theorem.

It is not difficult to show that in any free semigroup the strong quasi-order relation reduces to the identity relation. By Theorem 1, every free semigroup is embeddable in a generalized group. Since every semigroup is a homomorphic image of some free semigroup, but not every semigroup is embeddable in a generalized group, the class of semigroups embeddable in generalized groups does not satisfy the well-known criterion of G. Birkhoff[6] and cannot be characterized by a system of axioms each of which has the form of an equality.

Let us also note that the non-elementary condition (1) can be written in the form of a second-order axiom (the elementary axioms $\{A_n\}$ are first-order formulas).

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