MATHEMATICS

A. V. EFIMOV

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Abstract

Full Text

MATHEMATICS

A. V. EFIMOV

ON LINEAR METHODS OF SUMMATION OF FOURIER SERIES OF PERIODIC FUNCTIONS

(Presented by Academician A. N. Kolmogorov, 27 XI 1959)

1. With the aid of a triangular matrix

$$\Lambda = \{\lambda_k^{(n)}\} \quad (k = 0, 1, \dots, n+1;\ n = 0, 1, \dots;\ \lambda_0^{(n)} = 1,\ \lambda_{n+1}^{(n)} = 0) \qquad (1)$$

there is assigned to the Fourier series of a summable function $f(x)$ of period 2π the corresponding trigonometric polynomial $U_n(f, x, \Lambda)$:

$$U_n(f, x, \Lambda) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx) \quad (n = 0, 1, \dots),$$

where a_k and b_k are the Fourier coefficients of the function $f(x)$. By

$$K_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos kt$$

we shall denote the kernel of the method $U_n(f, x, \Lambda)$. S. M. Nikol' skii ([1]), B. Nad' ([2]), and Karamata and Tomić ([3]) gave conditions imposed on the matrix Λ, under which, for any summable function $f(x)$ of period 2π, at each of its Lebesgue points x the relation

$$\lim_{n\to\infty} U_n(f, x, \Lambda) = f(x). \qquad (2)$$

holds.

We give new conditions, which are an extension of the conditions given by S. M. Nikol' skii, B. Nad' , and Karamata and Tomić.

Theorem 1. For any matrix (1) the inequality

$$\int_0^\pi |K_n(t)|\,dt \le C_1 + C_2 \sum_{k=0}^{n-1} \frac{(k+1)(n-k)}{n+1} |\Delta^2 \lambda_k^{(n)}| + C_3 \sum_{k=0}^{n} \frac{|\lambda_k^{(n)}|}{n-k+1},$$

holds, where $\Delta^2\lambda_k^{(n)} = \lambda_k^{(n)} - 2\lambda_{k+1}^{(n)} + \lambda_{k+2}^{(n)}$, and C_1, C_2, C_3 are absolute constants.

Theorem 2. If the sequence (1) satisfies the condition

$$\sum_{k=0}^{n-1} \frac{(k+1)(n-k)}{n+1} |\Delta^2 \lambda_k^{(n)}| < C,$$

then, in order that for any summable function $f(x)$ of period 2π at each of its Lebesgue points x the relation (2) be satisfied, it is necessary and sufficient that the conditions

$$\lim_{n\to\infty} \lambda_k^{(n)} = 1 \quad (k = 1, 2, ...), \tag{3}$$

$$\sum_{k=0}^{n} \frac{|\lambda_k^{(n)}|}{n-k+1} < C \tag{4}$$

hold.

The necessity of condition (3) was proved by S. M. Nikol' skii ([1]), and the necessity of condition (4) by Sidon ([4]) (for the proof see ([5])).

II. Let $\omega(\delta)$ be a positive function that is a modulus of continuity ([6]). By $H[\omega]$ we denote the class of continuous functions $f(x)$ of period 2π whose modulus of continuity $\omega(\delta, f)$ satisfies the condition $\omega(\delta, f) \le \omega(\delta)$, and the class of conjugate functions corresponding to them is denoted by $\bar{H}[\omega]$. By $W^r_\beta H[\omega]$ we denote the class of functions $f(x)$ representable in the form of the series

$$f(x) = \frac{a_0}{2} + \sum_{k=0}^{\infty} \frac{1}{\pi k^r} \int_{-\pi}^{\pi} \varphi(x+t) \cos\left(kt + \frac{\beta\pi}{2}\right) dt \qquad (r \ge 0),$$

where $\varphi(x) \in H[\omega]$ and

$$\int_{-\pi}^{\pi} \varphi(x)\,dx = 0$$

(cf. ([7],[8])).

We study the asymptotic behavior of the quantity

$$\mathcal{E}_{U_n}(W^r_\beta H[\omega]) = \sup_{f\in W^r_\beta H[\omega]} \|f(x) - U_n(f,x,\Lambda)\|_{C_{2\pi}}.$$

An asymptotically exact law of decrease of the quantity $\mathcal{E}_{U_n}(\mathfrak{M})$ for some classes of functions $\mathfrak{M}$ and concrete approximation methods has been given in works by a number of authors (see, for example, ([9–15])). Put

$$C_1^{(n)}[\omega] = \sup_{f\in H[\omega]} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos nx\, dx \right|,$$

$$d_n[\omega] = \sup_{\substack{\varphi\in H[\omega]\\ \varphi(-t)=-\varphi(t)}} \left| \int_0^{1/n} \frac{\varphi(t)}{t}\, dt \right|, \qquad d_{n,k}[\omega] = \sup_{\substack{\varphi\in H[\omega]\\ \varphi(-t)=-\varphi(t)}} \left| \int_{1/n}^{1/(k+1)} \frac{\varphi(t)}{t}\, dt \right|,$$

$$h[n,\omega] = \omega\left(\frac{1}{n}\right) + \sum_{k=0}^{n-1} \frac{(k+1)(n-k)}{n+1}\, \omega\left(\frac{1}{k+1}\right) \left|\Delta^2 \lambda_k^{(n)}\right|.$$

If $\omega(\delta)$ satisfies the condition

$$\frac{1}{2}[\omega(\delta_1) + \omega(\delta_2)] \le \omega\left(\frac{\delta_1+\delta_2}{2}\right) \qquad (0 \le \delta_1 \le \delta_2), \tag{5}$$

then ([16])

$$C_1^{(n)}[\omega] = \frac{2}{\pi} \int_0^{\pi/2} \omega\left(\frac{2z}{n}\right) \sin z\, dz$$

$$d_n[\omega] = \frac{1}{2} \int_0^{1/n} \frac{\omega(2z)}{z}\, dz, \qquad d_{n,k}[\omega] = \frac{1}{2} \int_{1/n}^{1/(k+1)} \frac{\omega(2z)}{z}\, dz,$$

and for arbitrary $\omega(\delta)$ we have proved that

$$C_1^{(n)}[\omega] = \frac{2\theta}{\pi} \int_0^{\pi/2} \omega\left(\frac{2z}{n}\right) \sin z\, dz,$$

$$d_n[\omega] = \frac{\theta}{2} \int_0^{1/n} \frac{\omega(2z)}{z}\, dz, \qquad d_{n,k}[\omega] = \frac{\theta}{2} \int_{1/n}^{1/(k+1)} \frac{\omega(2z)}{z}\, dz,$$

where $2/3 \le \theta \le 1$. The constant $2/3$ cannot be improved, since it is attained if $\omega(\delta)$ is the Cantor step function.

Theorem 3. For any matrix Λ the inequalities

$$\mathcal{E}_{U_n}(W^0_\beta H[\omega]) \le \frac{C_1^{(n)}[\omega]}{\pi} \sum_{k=\nu+3}^{n} \frac{|\lambda_k^{(n)}|}{n-k+1} + \frac{2|\sin\frac{1}{2}\beta\delta|}{\pi}\, d_n[\omega] +$$

$$+\frac{2|\cos\frac{1}{2}\beta\pi|}{\pi} \sum_{k=0}^{\nu} |\Delta\lambda_k^{(n)}| \cdot \omega\left(\frac{1}{k+1}\right) +$$

$$+\frac{2|\sin\frac{1}{2}\beta\pi|}{\pi} \sum_{k=0}^{\nu-1} (k+1)|\Delta^2\lambda_k^{(n)}|\, d_{n,k}[\omega] + O(h[n,\omega]), \tag{6}$$

and for any $r > 0$

$$\mathcal{E}_{U_n}(W^r_\beta H[\omega]) \le \frac{C_1^{(n)}[\omega]}{\pi n^r} \sum_{k=\nu+3}^{n} \frac{|\lambda_k^{(n)}|}{n-k+1} +$$

$$+\frac{2|\cos\frac{1}{2}\beta\pi|}{\pi} \sum_{k=0}^{\nu} |\Delta\mu_k^{(n)}| \cdot \omega\left(\frac{1}{k+1}\right) + \frac{2|\sin\frac{1}{2}\beta\pi|}{\pi} \sum_{k=0}^{\nu-1} (k+1)|\Delta^2\mu_k^{(n)}|\, d_{n,k}[\omega] +$$

$$+O\left(\sum_{k=0}^{\nu} (k+1)|\Delta^2\mu_k^{(n)}|\omega\left(\frac{1}{k+1}\right)\right) + O\left(\frac{1}{n^r}\omega\left(\frac{1}{n}\right) \sum_{k=\nu+1}^{n-1} (n-k)|\Delta^2\lambda_k^{(n)}|\right), \tag{7}$$

where $\nu = [\frac{n+1}{2}]$; $\mu_0^{(n)} = 0$; $\mu_k^{(n)} = \dfrac{1-\lambda_k^{(n)}}{k^r}$ $(k = 1, 2, \dots, n+1)$; $\Delta^2\mu_k^{(n)} = \Delta\mu_k^{(n)} - \Delta\mu_{k+1}^{(n)} = \mu_k^{(n)} - 2\mu_{k+1}^{(n)} + \mu_{k+2}^{(n)}$.

In the class of all linear methods and for arbitrary moduli of continuity $\omega(\delta)$, inequalities (6) and (7) cannot be improved, since there exist linear methods for which these inequalities turn into asymptotic equalities.

Theorem 4. If the sequence (1) satisfies the conditions

$$\lambda_k^{(n)} \ge 0 \quad \text{or} \quad \lambda_k^{(n)} \le 0 \quad \text{for all } k = \nu+3, \dots, n, \tag{8}$$

then, for any modulus of continuity satisfying the condition

$$\delta \int_\delta^1 \frac{\omega(z)}{z^2}\, dz = O(\omega(\delta)),$$

the asymptotic equality

$$\mathcal{E}_{U_n}(H[\omega]) = \frac{C_1^{(n)}[\omega]}{\pi} \left| \sum_{k=\nu+3}^{n} \frac{\lambda_k^{(n)}}{n-k+1} \right| + O(h[n,\omega]), \tag{9}$$

holds; and for any $\omega(\delta)$ satisfying the condition

$$\int_0^\delta \frac{\omega(z)}{z}\,dz = O(\omega(\delta)),$$

the asymptotic equality

$$\mathcal{E}_{U_n}(\overline{H}[\omega]) = \frac{C_1^{(n)}[\omega]}{\pi} \left| \sum_{k=\nu+3}^{n} \frac{\lambda_k^{(n)}}{n-k+1} \right| + O(h[n,\omega]) \tag{10}$$

holds.

If, in addition to (8), the sequence (1) satisfies the conditions

$$\Delta\lambda_k^{(n)} \geq 0 \quad \text{or} \quad \Delta\lambda_k^{(n)} \leq 0 \quad \text{for } k = 0, 1, \dots, \nu, \tag{11}$$

then, for any modulus of continuity satisfying the conditions (5) and

$$\delta \int_0^1 \frac{\omega(z)}{z^2}\,dz \neq O(\omega(\delta)),$$

the equality

$$\mathcal{E}_{U_n}(H[\omega]) = \gamma_n[\omega, \Lambda] \left\{ \frac{C_1^{(n)}[\omega]}{\pi} \left| \sum_{k=\nu+3}^{n} \frac{\lambda_k^{(n)}}{n-k+1} \right| + \right.$$

$$\left. + \frac{2}{\pi} \left| \sum_{k=0}^{\nu} \Delta\lambda_k^{(n)} \omega\left(\frac{1}{k+1}\right) \right| \right\} + O(h[n,\omega]). \tag{12}$$

If the sequence (1) satisfies conditions (8) and $\Delta^2\lambda_k^{(n)} \geq 0$ $(k = 0, 1, \dots, \nu - 1)$, then for any modulus of continuity satisfying conditions (5) and

$$\int_0^\delta \frac{\omega(z)}{z}\,dz \neq O(\omega(\delta)),$$

the following equality holds:

$$\mathcal{E}_{U_n}[\overline{H}[\omega]] = \overline{\gamma}_n[\omega,\Lambda]\left\{\frac{C_1^{(n)}[\omega]}{\pi}\left|\sum_{k=\nu+3}^{n}\frac{\lambda_k^{(n)}}{n-k+1}\right| + \frac{2}{\pi}d_n[\omega] + \frac{2}{\pi}\sum_{k=0}^{\nu-1}(k+1)\Delta^2\lambda_k^{(n)}d_{n,k}[\omega]\right\} + O(h[n,\omega]), \tag{13}$$

where

$$\frac{1}{2} \le \gamma_n[\omega,\Lambda], \overline{\gamma}_n[\omega,\Lambda] \le 1,$$

and moreover $\gamma_n[\omega,\Lambda] = \overline{\gamma}_n[\omega,\Lambda] = 1$, if the matrix Λ is such that

$$\sum_{k=0}^{\nu}\left|\Delta\lambda_k^{(n)}\right|\omega\left(\frac{1}{k+1}\right) = O\left(\omega\left(\frac{1}{n}\right)\right)$$

$$\left(\sum_{k=0}^{\nu-1}(k+1)\left|\Delta^2\lambda_k^{(n)}\right|\int_{1/n}^{1/(k+1)}\frac{\omega(t)}{t}\,dt = O\left(\omega\left(\frac{1}{n}\right)\right)\right),$$

or, for the matrix Λ, the relation

$$\sum_{k=\nu+3}^{n}\frac{\left|\lambda_k^{(n)}\right|}{n-k+1} = O(1).$$

We note that if the system of numbers $\mu_0^{(n)} = 0$,

$$\mu_k^{(n)} = \frac{1-\lambda_k^{(n)}}{k^r} \quad (k = 1, 2, \dots, n+1)$$

satisfies the conditions

$$\Delta\mu_k^{(n)} \le 0, \qquad \Delta^2\mu_k^{(n)} \ge 0, \tag{14}$$

then, for

$$\delta\int_\delta^1\frac{\omega(z)}{z^2}\,dz = O\left(\int_0^\delta\frac{\omega(z)}{z}\,dz\right) = O(\omega(\delta)),$$

inequality (7) turns into an asymptotic equality, a particular case of which (r an integer ≥ 0, $\omega(\delta) = \delta^\alpha$, $0 \le \alpha < 1$) was obtained earlier by A. F. Timan ([12]). If, in addition to (14), the system of numbers

$$\eta_0^{(n)}(\varepsilon) = 0, \qquad \eta_k^{(n)}(\varepsilon) = \frac{1-\lambda_k^{(n)}}{k^{r+\varepsilon}} \quad (r \ge 0,\ k = 1, 2, \dots, n+1)$$

for some $\varepsilon > 0$ satisfies the conditions

$$\Delta\eta_k^{(n)}(\varepsilon) \le 0, \qquad \Delta^2\eta_k^{(n)}(\varepsilon) \ge 0, \tag{15}$$

then for any $\omega(\delta)$ inequality (7) turns into an asymptotic equality, while in equality (13) in this case $\gamma_n[\omega,\Lambda] = 1$, i.e. the method is "close" to the Fourier sums. We note that for $\varepsilon \geq 1$, from inequality (15), for any $r \geq 0$ the inequalities (14) follow; and then from equality (13), as a particular case with $\varepsilon = 1$, we obtain the result of I. M. Ganzburg $((^{15}), theorem 3)$.

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Note: Figure translations are in progress. See original paper for figures.