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# HYDROMECHANICS

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**Abstract**

**Full Text**

HYDROMECHANICS

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**ISOTHERMAL EXPANSION OF A GAS CLOUD**

*(Presented by Academician A. D. Sakharov, 25 XII 1959)*

1. For a number of astrophysical problems, the problem of the expansion of a gas cloud occurring under conditions of external irradiation is of interest. The law of conservation of energy in the cloud is then not satisfied, and the entropy of the gas particles increases as the degree of expansion increases. Generally speaking, expansion of this kind has a complicated character, but there exists a simple limiting case of non-isentropic expansion—*isothermal expansion*. In this case it is assumed that the gas temperature does not depend on the spatial coordinate, but depends in an arbitrary way on time, i.e.  $\theta = \theta(t)$ . The dependence of the temperature of the gas cloud on time is determined by the character of the external irradiation. The isothermal approximation has already been used earlier to obtain certain exact solutions of the one-dimensional gas-dynamic equations <sup>(1-4)</sup>.

In the present note we shall show that in the isothermal case there exists an asymptotic solution for the problem of expansion of a gas cloud. The forces of self-gravitation and the magnetic field will not be taken into account.

2. If we introduce the dimensionless variables  $\xi = x/l$ ,  $\tau = c_0 t/l$ ,  $u = v/c_0$ ,  $\delta = \rho/\rho_0$ , where  $l$  is the characteristic size of the cloud,  $\rho$  is the initial gas density in it,  $c_0$  is the isothermal speed of sound  $c_0 = \sqrt{A\theta}$  ( $A$  is the gas constant in the equation of state  $p = A\rho\theta$ ), and seek a particular solution of the one-dimensional gas-dynamic equations with  $\theta = \text{const}$  in the form proposed by A. D. Sakharov\*:

$$\delta = \frac{B}{[\xi_0(\tau)]^\nu} \Phi \left[ \frac{\xi}{\xi_0(\tau)} \right], \quad u = u_0(\tau) V \left[ \frac{\xi}{\xi_0(\tau)} \right], \quad (1)$$

$$\left( B = \frac{1}{\nu \int_0^\infty \Phi(\eta) \eta^{\nu-1} d\eta} \right); \text{ the total mass of the cloud, for example, in the plane case:}$$

$$M = 2\rho_0 l,$$

then we obtain:

$$\delta = \frac{1}{[\xi_0(\tau)]^\nu} \exp \left\{ - \left[ \frac{\nu\Gamma(\nu/2)}{2} \right]^{2/\nu} \left[ \frac{\xi}{\xi_0(\tau)} \right]^2 \right\},$$

$$u = \left[ 2^{\nu-1} \nu \Gamma\left(\frac{\nu}{2}\right) \right]^{1/\nu} \sqrt{\ln \xi_0(\tau)} \frac{\xi}{\xi_0(\tau)}. \quad (2)$$

The function  $\xi_0(\tau)$  in (2) is determined from the equation

$$\int_1^{\xi_0(\tau)} \frac{d\xi_0}{\sqrt{\ln \xi_0}} = \left[ 2^{\nu-1} \nu \Gamma\left(\frac{\nu}{2}\right) \right]^{1/\nu} \tau. \quad (3)$$

\* Private communication.

The parameter  $\nu$  is equal, respectively, to 1, 2, 3 for the cases of plane, cylindrical, and spherical symmetry.

The solution (2)–(3) corresponds to the initial conditions

$$\tau = 0, \quad u = 0, \quad \delta = \exp \left\{ - \left[ \frac{\xi^{\Gamma(\nu/2)}}{2} \right]^{2/\nu} \xi^2 \right\}. \quad (4)$$

A solution of the type (2), (3) is already known in the literature <sup>(1–3)</sup>. In the solution (2), (3) the velocity  $u$  depends linearly on the spatial coordinate  $\xi$ . A solution of this kind for isentropic motions of a gas was obtained and studied earlier in the works of L. I. Sedov and others (see, for example, <sup>(5)</sup>). In the isothermal case, the possibility of such a solution was noted earlier in <sup>(1)</sup>; the general form of the equations of state admitting such a solution was also investigated there. The solution (2), (3) for the cylindrical case is given in works <sup>(2,3)</sup>. It is easy to verify that passing to a gas-cloud temperature depending on time does not change formula (2), but leads, for determining the function  $\xi_0(\tau)$ , to a different equation than (3). For the cylindrical case this follows from <sup>(2,3)</sup>. The solution (2), (3) is very simply generalized to the case of a more general equation of state, admitted according to <sup>(1)</sup>, than the equation of state of an ideal gas.

3. An important feature of the solution (2), (3) is its asymptotic character for any initial distribution of density and velocity of motion in the gas. It is known that in the isentropic expansion of a gas an asymptotic solution does not exist <sup>(5,6)</sup>. In the case of isothermal expansion, the continuous supply to the gas of additional portions of energy at the expense of the energy of external radiation ultimately leads to the initial energy of the gas cloud becoming smaller than the kinetic energy of expansion. At this stage of expansion the initial conditions are naturally forgotten.

For the expansion of a plane gas layer, one can directly demonstrate the asymptotic character of the solution (2), (3), using the solution of isothermal expansion for an initial gas-density distribution in the form of a “bar.” This solution can be found by applying the hodograph transformation (6), reducing the system of gas-dynamic equations to an equation of telegraph type, for which the Cauchy problem is solved using the known Riemann function (7). Here we give only the result of the solution:

$$\begin{aligned}
 z(\alpha, \beta) = & e^{2\alpha} \int_{-2\alpha}^0 e^x I_0(x) dx + e^{2\alpha} \int_{-2\beta}^0 e^y I_0(y) dy \\
 & - 2 \int_{\alpha}^{\beta} e^{2x} J_0(2\sqrt{-(x-\alpha)(x-\beta)}) dx \\
 & + \int_{\alpha}^{\beta} (\alpha - \beta) e^{2x} \left( \int_{-2x}^0 e^y I_0(y) dy \right) J_1(2\sqrt{-(x-\alpha)(x-\beta)}) \frac{1}{\sqrt{-(x-\alpha)(x-\beta)}} dx,
 \end{aligned} \tag{5}$$

where

$$\begin{aligned}
 \xi = & \frac{1}{2} e^{-\alpha-\beta} \left\{ \left[ (\alpha - \beta) - \frac{1}{2} \right] \frac{\partial z}{\partial \alpha} + \left[ (\alpha - \beta) + \frac{1}{2} \right] \frac{\partial z}{\partial \beta} - 2z(\alpha - \beta) \right\}, \\
 t = & \frac{1}{4} e^{-\alpha-\beta} \left( \frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta} - 2z \right), \\
 u = & 2(\alpha - \beta), \quad \ln \delta = 2(\alpha + \beta).
 \end{aligned} \tag{6}$$

In the solution for the initial gas-density distribution in the form of a “bar,” for  $\tau \gg 1$  two regions are obtained: the region of the incident isothermal rarefaction wave and the region of the wave reflected from the center of the layer. The solution for the region of the reflected wave is given above (5), (6),

This solution has a simple form at the center of the layer, where  $\alpha = \beta$ . From (5), (6) we obtain

$$\sqrt{\frac{1}{\delta}} I_0\left(\frac{1}{2} \ln \frac{1}{\delta}\right) = \tau, \quad \tau \geq 1. \tag{7}$$

Formula (7) was also obtained in (6) (p. 168).

Let us prove the asymptotic agreement of  $\delta(\tau)$  from (7) with  $\delta(0, \tau)$  from (2). Using the asymptotic form of the Bessel function  $I_0(x) = e^x / \sqrt{2\pi x}$ , from (7) we obtain

$$\frac{1}{\delta} \frac{1}{\sqrt{-\ln \delta}} \simeq \sqrt{\pi} \tau. \tag{7'}$$

On the other hand, for  $\tau \rightarrow \infty$  and  $\nu = 1$ , instead of (3) we have

$$\frac{\xi_0(\tau)}{\sqrt{\ln \xi_0(\tau)}} \simeq \sqrt{\pi\tau}. \quad (3')$$

Substituting into (3')  $\xi_0(\tau) = 1/\delta(0, \tau)$  according to (2), we obtain complete agreement of (3') with (7').

The regions of the incident rarefaction wave and of the wave reflected from the center of the layer are separated by a weak discontinuity propagating in the direction of gas motion with the sound speed  $c_0$ . The coordinate  $\xi$  of this weak discontinuity, as well as the values of  $\delta, u$  on it, have the form

$$\xi = \tau(2 \ln \tau - 1) + 1, \quad \delta = \frac{1}{\tau^2}, \quad u = 2 \ln \tau. \quad (8)$$

In (8),  $\xi = 0$  corresponds to the center of the layer. Let us next consider the asymptotic form of (2) at this boundary, substituting there  $\xi(\tau)$  from (8):

$$\delta = \frac{1}{\xi_0(\tau)} \exp \left\{ -\frac{\pi}{4} \left[ \frac{\tau(2 \ln \tau - 1) + 1}{\xi_0(\tau)} \right]^3 \right\}.$$

Using (3'), we pass to the limit  $\tau \rightarrow \infty$ ; we obtain:

$$\delta \simeq \frac{1}{\xi_0(\tau)} \exp \left[ -\frac{\ln^2 \tau}{\ln \xi_0(\tau)} \right] \simeq \frac{\pi \ln \xi_0(\tau)}{\xi_0^2(\tau)} \simeq \frac{1}{\tau^2},$$

i.e., on boundary (8) the values of the gas density in both solutions asymptotically coincide. One can also prove the agreement of the gas velocities, which in both solutions behave asymptotically as  $u = \xi/\tau$ . From the fact that the solution of the isothermal expansion for a "slab" coincides with solution (2), (3) at the center of the layer and at boundary (8), their asymptotic agreement throughout the entire region of the reflected wave strictly follows. The character of the establishment of the asymptotic regime can be determined by a detailed comparison of solution (2), (3) with solution (5), (6) throughout the entire region of the reflected wave. In the region of the incident rarefaction wave, asymptotic agreement of the two solutions is not obtained. This is evident from the fact that, in the incident rarefaction wave, the gas-density profile at each instant of time is  $\tau\delta \sim e^{-\alpha\xi}$ , whereas in solution (2), (3) this profile is  $\delta \sim e^{-b\xi^2}$ . However, as  $\tau \rightarrow \infty$ , the amount of matter in this region becomes negligibly small in comparison with the amount of matter in the region of the reflected wave. From (8) it is easy to find that

$$\frac{M(\tau)}{M_0} = \frac{1}{\tau}, \quad \tau \geq 1,$$

where  $M(\tau)/M_0$  is the relative amount of matter in the region of the incident wave.

Thus, the region bearing the imprint of the initial conditions disappears as  $\tau \rightarrow \infty$ , while the region of the reflected wave asymptotically coincides with (2), (3).

We assume that the solution (2), (3) is also asymptotic for any initial distributions of the density and velocity of motion of the gas. The cylindrical and spherical cases, naturally, do not differ in this respect from the plane case.

4. In conclusion, let us write down the criterion for the applicability of the formulas of isothermal expansion, taking into account only the processes of so-called true absorption of radiation in the gas <sup>(8)</sup>:

$$\frac{1}{\beta} > \frac{l}{\lambda} > \beta, \quad (9)$$

where

$$\beta = \frac{1}{\sigma c} \frac{\rho_0 c_0^3}{\theta}.$$

The temperature of the gas cloud  $\theta$  is related to the temperature of the radiating surfaces  $T$  by means of the dilution factor  $W$  ( $\theta^4 = WT^4$ ) <sup>(8)</sup>.

In (9),  $\lambda$  for  $l/\lambda < 1$  is defined as the mean free path for the absorption of photons in the expanding cloud with respect to the spectrum of the external radiation; for  $l/\lambda > 1$ ,  $\lambda$  is the mean Rosseland absorption path of radiation in the cloud at density  $\rho_0$  and gas temperature  $\theta$ .

Criterion (9) follows from the simple consideration that the temperature difference between the edge and the center of the cloud must be small. This also means that the fraction of the radiant-energy flux absorbed in the expanding cloud and converted into kinetic energy must be small in comparison with the radiant-energy flux passing through the cloud. The kinetic energy of the cloud can easily be found from (2), while the radiant-energy flux passing through the cloud is estimated from elementary considerations separately for two limiting cases: optically thick and optically thin clouds. In the case of an optically thin cloud, in deriving (9) it is also necessary to formulate an additional condition ensuring the actual absorption of a fraction of the radiant energy equal to the kinetic energy of the cloud's expansion.

When criterion (9) is not satisfied, the formulas of isothermal expansion may be useful in the sense that they determine the most rapid expansion for a given initial gas density in the cloud  $\rho_0$  and initial temperature  $\theta$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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