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Abstract

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POSITIVE-DEFINITE FUNCTIONS AND HOMOGENEOUS RANDOM FIELDS ON GROUPS AND HOMOGENEOUS SPACES

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1. It is known that the spectral theory of stationary random processes is a consequence of Bochner' s theorem on the general form of positive-definite functions on the line. The generalization of Bochner' s theorem, due to A. Weil ([1]) and D. A. Raikov ([2]), to the case of positive-definite functions on an arbitrary commutative locally compact group made it possible without difficulty to extend the spectral theory of stationary processes to a broader class of homogeneous random fields on such groups (see ([3])). However, the very definition of a homogeneous random field on a group is in no way connected with its commutativity, and only the absence of explicit formulas giving the general form of positive-definite functions on sufficiently broad classes of noncommutative groups*, until very recently, hindered the construction of a spectral theory of homogeneous fields on noncommutative groups. At the same time it should be borne in mind that, by virtue of a general theorem of I. M. Gelfand and D. A. Raikov ([5]), the description of all positive-definite functions on any locally compact group is directly reduced to the description of all its unitary representations; proceeding from this, the problem of finding an analogue of Bochner' s theorem for noncommutative groups can easily be formulated as a certain problem in the theory of unitary representations of groups, which is also of considerable independent interest for this theory. In the very recent period, the intensive development of the theory of unitary representations has led to the appearance of works ([6–8]) which finally make it possible to formulate the generalization of Bochner' s theorem needed for the theory of homogeneous fields to the case of positive-definite functions on a sufficiently broad class of "good" noncommutative groups, and thereby have made possible a further substantial generalization of the classical spectral theory of stationary random processes. The present note is devoted to the exposition of these generalizations.

2. In what follows we shall consider only separable locally compact groups of type I (cf. ([6],[8])); for brevity such groups will be called simply groups of type I. Recall that among such groups of type I are all compact topological groups and all commutative locally compact groups, as well as all connected semisimple Lie groups ([9]) and all algebraic Lie groups ([10]).

Denote by $\widehat{G}$ the "dual object" of a group G of type I, i.e. the totality of all possible equivalence classes of irreducible unitary representations of this group. According to Mackey ([6]), on $\widehat{G}$ a "natural Borel structure" can be defined— the Borel field $\mathfrak{A}$ of "measurable" sets $\Lambda \subset \widehat{G}$. Let $\{T^{(\lambda)}(g)\}$ be one of the representations of the group G corresponding to the point $\lambda \in \widehat{G}$ and acting in the Hilbert space—

* An exception is the comparatively narrow class of compact groups, for which the general form of positive-definite functions was established already in 1941 by Bochner ([4]).

the space $H^{(\lambda)}$. We identify with one another all spaces $H^{(\lambda)}$ having the same dimension; then $\widehat{G}$ decomposes into the sum of disjoint sets $\widehat{G}_n,\ n = \infty, 1, 2, ...,$ corresponding to classes of equivalent n-dimensional unitary irreducible representations acting in the same n-dimensional unitary space H_n. Let now $F(\Lambda)$ be an "operator measure" on $\widehat{G}$—a countably additive function on $\mathfrak{A}$, whose values for $\Lambda \subset \widehat{G}_n$ are Hermitian nonnegative linear operators in H_n. Suppose that there exists the integral

$$B(g) = \int_{\widehat{G}} \mathrm{Tr}\{T^{(\lambda)}(g)\, F(d\lambda)\}, \tag{1}$$

where $\mathrm{Tr}\, A$ denotes the trace of the operator A, and the integral over $\widehat{G}$ (as everywhere below) is understood as the sum of integrals over all $\widehat{G}_n \subset G$. It is easy to see that the function $B(g)$ will be positive-definite on G, i.e. such that

$$\sum_{i,k=1}^{N} B(g_k^{-1} g_i) a_i \bar{a}_k \geq 0$$

for every positive integer N, any complex $a_1, a_2, \dots, a_N$, and any $g_1, g_2, \dots, g_N \in G$. The generalization of Bochner' s theorem to "good" noncommutative groups, of which we spoke at the end of § 1, consists precisely in the fact that for groups G of type I the functions of the form (1) exhaust all continuous positive-definite functions on G; thus the following holds:

Theorem 1. *Every continuous positive-definite function on a group G of type I admits a representation of the form* (1), *where $F(\Lambda)$ is an "operator measure" on $\widehat{G}$ such that* $\mathrm{Tr}\, F(\widehat{G}) < \infty$.

In the special case when the group G is commutative, $T^{(\lambda)}(g) \equiv \chi^{(\lambda)}(g)$, and Theorem 1 becomes the Weil–Raikov theorem (which also includes the classical Bochner theorem). If the group G is bicompact, then all representations $T^{(\lambda)}(g)$ are finite-dimensional, and $\widehat{G}$ is discrete; in this case Theorem 1 passes into Theorem 6 of the work ([4]).

3. We shall call a **random field on the group** G a function $\xi(g)$ with values in a Hilbert space $\mathfrak{H}$ of complex random variables with zero mathematical expectation and bounded variance, continuous in the sense of the strong topology in $\mathfrak{H}$. The field $\xi(g)$ will be called **homogeneous** if

$$\mathbf{M}\xi(g_1)\overline{\xi(g_2)} = \mathbf{M}\xi(gg_1)\overline{\xi(gg_2)} \quad \text{for every } g \in G, \tag{2}$$

where $\mathbf{M}$ denotes mathematical expectation. In this case

$$\mathbf{M}\xi(g_1)\overline{\xi(g_2)} = B(g_2^{-1}g_1),$$

where $B(g)$ is a positive-definite function on G, called the correlation function of the field $\xi(g)$; for groups G of type I the general form of this function is given by formula (1). Let now $Z(f_1, f_2)$ be a random function (with $\mathbf{M}Z(f_1, f_2) \equiv 0$), linearly dependent on the vector f_1 of the Hilbert space H and antilinearly* dependent on $f_2 \in H$; in this case we shall write $Z(f_1, f_2) = (Zf_1, f_2)$, where Z is a **random linear operator** in H. Denote by $\mathfrak{B}(H)$ the linear space of all random linear operators in H, and define an "operator random measure" on $\hat{G}$ as a countably additive function $Z(\Lambda)$ on $\mathfrak{A}$ with values $Z(\Lambda) \in \mathfrak{B}(H_n)$ for $\Lambda \subset \hat{G}_n$. Let $Z(\Lambda)$ be an operator random measure satisfying the following "orthogonality condition" :

$$\mathbf{M}\left(Z(\Lambda_1)f_1, f_2\right)\overline{\left(Z(\Lambda_2)g_1, g_2\right)} = (f_1, g_1)\left(F(\Lambda_1 \cap \Lambda_2)g_2, f_2\right), \tag{3}$$

* The function $Z(f)$ is called antilinear if $Z(\lambda f_1 + \mu f_2) = \bar{\lambda}Z(f_1) + \bar{\mu}Z(f_2)$.

where $F(\Lambda)$, for $\Lambda \subset \hat{G}_n$, is an operator in H_n defining an ordinary operator-valued measure on $\hat{G}$. If the integral

$$\xi(g) = \int_{\hat{G}} \mathrm{Tr}\{T^{(\lambda)}(g)Z(d\lambda)\} \tag{4}$$

(defined on the basis of strong convergence in the space $\mathfrak{H}$) is meaningful, then, obviously, $M\xi(g_1)\overline{\xi(g_2)} = B(g_2^{-1}g_1)$, where $B(g)$ is the function (1), i.e. the field $\xi(g)$ will be homogeneous. The generalization of the spectral theory of stationary processes to homogeneous fields on "good" topological groups consists in the fact that, for groups G of type I, formula (4) exhausts all possible homogeneous random fields on G, so that the following holds:

Theorem 2. *Every homogeneous random field $\xi(g)$ on a group G of type I admits a representation (4), where $Z(\Lambda)$ is a random operator-valued measure satisfying the orthogonality condition (3).*

4. Let now $X = G/K$ be a topological homogeneous space (G is a topological group, K its closed bicompact subgroup). A random field $\xi(x)$ on X we define as a continuous mapping of X into $\mathfrak{H}$. The field $\xi(x)$ is called homogeneous if its correlation function $B(x_1, x_2) = M\xi(x_1)\overline{\xi(x_2)}$ satisfies the relation

$$B(x_1, x_2) = B(gx_1, gx_2) \quad \text{for any } g \in G. \tag{5}$$

The question of the general form of the correlation functions $B(x_1, x_2)$ of homogeneous fields is, obviously, equivalent to the question of the general form of positive-definite kernels $B(x_1, x_2)$ satisfying (5). With the aid of the Gelfand-Raikov theorem ([5]) this question is not difficult to reduce to a certain question in representation theory (cf. ([11–13])). Denote by $H^{(\lambda)}(K)$ the maximal subspace of the space $H^{(\lambda)}$ invariant with respect to all transformations $T^{(\lambda)}(k)$, $k \in K$, and by $P^{(\lambda)}(K)$ the projection operator in $H^{(\lambda)}$ onto $H^{(\lambda)}(K)$; let $\hat{G}_n(K)$, $n = \infty, 1, 2, \ldots$, be the set of all $\lambda \in \hat{G}$ for which $H^{(\lambda)}(K)$ is an n-dimensional space $H_n(K)$ (which we shall regard as one and the same for all $\lambda \in \hat{G}_n(K)$), and let $F_K(\Lambda)$ be an operator-valued measure on $\hat{G}(K) = \bigcup_n \hat{G}_n(K)$, whose values for $\Lambda \subset \hat{G}_n(K)$ are nonnegative-definite linear operators in $H_n(K)$. If now g_1 and g_2 are two arbitrary elements of G from the cosets modulo K defining the points x_1 and x_2, then it is not difficult to verify that the integral

$$B(x_1, x_2) = \int_{\hat{G}(K)} \mathrm{Tr}\{P^{(\lambda)}(K)T^{(\lambda)}(g_2^{-1}g_1)P^{(\lambda)}(K)F_K(d\lambda)\} \tag{6}$$

(if it exists) will depend only on x_1 and x_2 (and not on g_1 and g_2) and will be an invariant positive-definite kernel on X. For spaces X with a motion group G of type I one can show that the functions (6) exhaust all continuous positive-definite invariant kernels, so that the following holds:

Theorem 3. *Every continuous positive-definite invariant kernel on a homogeneous space X with a motion group G of type I admits a representation of the form (6).*

If in each of the $H_n(K)$ we choose a definite basis, then formula (6) is rewritten in the form of an integral expansion of the kernel $B(x_1, x_2)$ in all possible sets of linearly independent zonal spherical functions $\{\Phi_{ij}^{(\lambda)}(x_1, x_2), \lambda \in \hat{G}(K)\}$ of the space X. Let us also recall that, by virtue of ([13]), representation (6) also holds for all invariant positive-definite kernels on symmetric homogeneous spaces X with that

only simplification that here $\hat{G}(K) \equiv \hat{G}_1(K)$ (i.e., to each $\lambda \in \hat{G}$ here there corresponds not more than one zonal spherical function $\Phi^{(\lambda)}(x_1, x_2)$).

5. We shall write a random function $Z(f_1, \bar{f}_2)$ (with zero mathematical expectation), depending linearly on the vector $f_1 \in H^{(1)}$ and antilinearly on

the vector $f_2 \in H^{(2)}$, in the form (Zf_1, f_2), where Z is a random linear operator from $H^{(1)}$ into $H^{(2)}$. Let $\hat{G}_{n,m}(K) = \hat{G}_n \cap \hat{G}_m(K)$, $m \leq n$, and consider a random operator measure $Z(\Lambda)$ on $\hat{G}(K)$—a countably additive function of $\Lambda = \Lambda_0 \cap \hat{G}(K)$, $\Lambda_0 \in \mathfrak{A}$, whose values for $\Lambda \subset \hat{G}_{n,m}(K)$ are random linear operators from H_n into $H_m(K)$. If the measure $Z(\Lambda)$ is such that

$$\mathrm{M}\,(Z(\Lambda_1)f_1, f_2)\,\overline{(Z(\Lambda_2)g_1, g_2)} = (f_1, g_1)(F_K(\Lambda_1 \cap \Lambda_2)g_2, f_2), \tag{7}$$

where $F_K(\Lambda)$ is an operator measure on $\hat{G}(K)$, and the integral exists

$$\xi(x) = \int_{\hat{G}(K)} \mathrm{Tr}\{Z(d\lambda)T^{(\lambda)}(g)P^{(\lambda)}(K)\}, \tag{8}$$

then $\xi(x)$, as is easy to see, will depend only on the point $x \in X$ determined by the element $g \in G$, and will be a homogeneous random field on X. In this connection the following holds.

Theorem 4. *Every homogeneous random field on a homogeneous space X with motion group G of type I admits a representation of the form (8). The same representation is also admitted by all homogeneous random fields on symmetric homogeneous spaces; moreover, in this case the space $H^{(\lambda)}(K)$ is one-dimensional for all $\lambda \in \hat{G}(K)$, i.e. $Z(\Lambda)$ for every Λ is a random linear functional.*

After passage to coordinate notation, formula (8) turns into the integral expansion of homogeneous random fields on X in all possible spherical functions of this space.

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Note: Figure translations are in progress. See original paper for figures.