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Abstract

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ON A METHOD FOR SOLVING THE POINCARÉ PROBLEM FOR ANALYTIC FUNCTIONS

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We shall solve the following boundary-value problem: find a function $f(z)$, analytic in the unit disk $|z| < 1$, which on its boundary Γ satisfies the condition

$$\operatorname{Re}\{a(z)f'(z) + b(z)f(z)\}\big|_{\Gamma} = c(z); \tag{1}$$

$a(z), b(z), c(z)$ are functions given on Γ, satisfying a Hölder condition with exponent ν $(0 < \nu \leqslant 1)$, $|a(z)| \neq 0$.

Problem (1) was considered by I. N. Vekua (^{1}), and its solution was equivalently reduced to the solution of a certain singular integral equation.

Denote by $n = \frac{1}{2\pi}\Delta[\arg a(t)]$ the index of problem (1) (an integer equal to the ratio of the increment of the function $\arg a(t)$ under one counterclockwise traversal by the point t of the contour Γ to 2π). Then, by multiplication by a real function, problem (1) can be brought to the form

$$\operatorname{Re}\{z^n e^{p(z)} f'(z) + \gamma(z)f(z)\}\big|_{\Gamma} = c_1(z). \tag{2}$$

Here $p(z)$ is analytic inside the unit disk $|z| < 1$,

$$\operatorname{Im} p(z)\big|_{\Gamma} = \arg\frac{a(z)}{z^n}, \qquad \gamma(z) = b(z)e^{\operatorname{Re} p(z)}, \qquad c_1(z) = c(z)e^{\operatorname{Re} p(z)}. \tag{3}$$

We shall assume that the expansion of the function $\gamma(z)$ in a Fourier series on the unit circle contains a finite number of terms with negative powers, i.e.

$$\gamma(z) = \sum_{\mu=-m}^{\infty} \gamma_\mu e^{i\mu\varphi}, \qquad \gamma_{-m} \neq 0. \tag{4}$$

This condition is satisfied, for example, by functions $\gamma(z)$ that are polynomials of finite degree in x and y.

When this condition (4) is fulfilled, our problem can be written in the form

$$\operatorname{Re}\left\{\frac{z^k e^{p(z)} f'(z) + Q(z) f(z)}{z^l}\right\} = c_1(z) \tag{5}$$

with $k \geqslant 0,\ l \geqslant 0$; $Q(z)$ is analytic and $Q(0) \neq 0$ for $k > 0$. The numbers k, l and the function $Q(z)$ are expressed in the following way in terms of m, n, and $\gamma(z)$:

$$\begin{array}{llll} 1.\ k = m + n, & l = m, & Q(z) = z^m \gamma(z) & \text{for } m \geqslant 0 \text{ and } m \geqslant -n. \\ 2.\ k = 0, & l = -n, & Q(z) = z^{-n} \gamma(z) & \text{for } m \geqslant 0 \text{ and } m < -n. \\ 3.\ k = n, & l = 0, & Q(z) = \gamma(z) & \text{for } m < 0 \text{ and } n \geqslant 0. \\ 4.\ k = 0, & l = -n, & Q(z) = z^{-n} \gamma(z) & \text{for } m < 0 \text{ and } n \leqslant 0. \end{array} \tag{6}$$

Remark 1. $k > 0$ and $Q(0) = 0$ can hold simultaneously only in the third case.

Solving problem (5) with respect to the analytic function $z^k e^{p(z)} f'(z) + Q(z) f(z)$, we obtain an ordinary differential equation equivalent to our problem, with a singular point $z = 0$:

$$z^k e^{p(z)} f'(z) + Q(z) f(z) = \frac{z^l}{2\pi} \int_\Gamma c_1(t) \frac{t + z}{t - z}\, ds +$$

$$+ a_0 + a_1 z + \cdots + a_{lz}^{l} - \overline{a}_{l-1} z^{l+1} - \cdots - \overline{a}_0 z^{2l}; \tag{7}$$

$a_0, a_1, \ldots, a_{l-1}$ are arbitrary complex numbers, a_l is a purely imaginary number.

In the case $k = 0$, equation (7) has no singularities; therefore our problem (1) is always solvable, and the homogeneous problem has a nontrivial solution depending on $2l + 3$ real parameters.

For $k > 0$ we transform equation (7) into a form more convenient for us. To this end, we divide both sides of the equation by $e^{p(z)}$ and make the substitution

$$f(z) = f_1(z) \exp\left[-\int_0^z \Phi(t)\, dt\right],$$

where $\Phi(z)$ is defined as follows. Let

$$e^{-p(z)} Q(z) = M_0 + M_1 z + \cdots + M_{kz}^{k} + \cdots;$$

then

$$\Phi(z) = M_k + M_{k+1} z + \cdots + M_{k+\mu} z^\mu + \cdots.$$

Our equation (7) takes the form

$$z^k f_1'(z) + \left(M_0 + M_1 z + \dots + M_{k-1} z^{k-1}\right) f_1(z) =$$

$$= \exp\left[p(z) + \int_0^z \Phi(t)\,dt\right]\left[\frac{1}{2\pi}\int_\Gamma c_1(t)\frac{t+z}{t-z}\,ds + a_0 + \dots + a_{lz}^l - \dots - \overline{a}_0 z^{2l}\right]$$

$$= e^{F(z)}[P(z) + S_{2l}(z)]. \tag{8}$$

In the case $k = 1$, the equation

$$z f_1'(z) + M_0 f_1(z) = e^{F(z)}[P(z) + S_{2l}(z)] \tag{9}$$

has a unique regular solution if $M_0 \neq -n$, where n is a natural number.

In the case $M_0 = -n$, for the solvability of the problem it is necessary and sufficient that the condition

$$\int_\Gamma e^{F(z)}\left[P(z) + (a_0 + a_1 z + \dots + a_{l-1} z^{l-1} +\right.$$

$$\left. + a_{lz}^l - \overline{a}_{l-1} z^{l+1} - \dots - \overline{a}_0 z^{2l})\right]\frac{dz}{z^{n+1}} = 0 \tag{10}$$

be satisfied.

This condition represents two real equations with $2l + 1$ unknowns, and the homogeneous equation (9) has a nontrivial solution.

Thus, the solution of problem (1) has been reduced to solving the system (10) and equation (9).

In the case $k > 1$, $M_0 \neq 0$, for the solvability of equation (8) it is necessary and sufficient that the following conditions be satisfied:

$$\int_{l_j} \frac{P(z) + (a_0 + \dots + a_{lz}^l - \dots - \overline{a}_0 z^{2l})}{z^{k - M_{k-1}}} \times$$

$$\times \exp\left[F(z) - \left(\frac{M_0}{k-1}\frac{1}{z^{k-1}} + \dots + \frac{M_{k-2}}{z}\right)\right] dz = 0; \tag{11}$$

l_j is the contour consisting of the segment $z = re^{i\frac{2\pi j}{k-1}}$ $(0 \leqslant r \leqslant 1)$, then the arc $z = e^{i\varphi}$ $\left(\frac{2\pi j}{k-1} \leqslant \varphi \leqslant \frac{2\pi(j+1)}{k-1}\right)$, and the segment $z = re^{i\frac{2\pi(j+1)}{k-1}}$ $(0 \leqslant r \leqslant 1)$, traversed in the reverse direction.

When conditions (11) are satisfied, the differential equation (8) has a unique solution regular in the disk $|z| < 1$. The solvability conditions (11) for equation (8) were given by Horn[2].

Thus, in the case $k > 1$, the question of solvability of problem (1) reduces to the question of solvability of the linear system (11). The linear system (11) consists of $2(k-1)$ real equations with $2l+1$ unknowns.

If $M_0 = 0$, $k > 0$, then, according to Remark 1, case 3 of formulas (6) occurs.

Let $q = \min\{n, -m\}$, $q > 0$; solving problem (5), we obtain the following problem:

$$z^k e^{p(z)} f'(z) + Q(z) f(z) = \frac{1}{2\pi} \int_\Gamma c_1(t) \frac{t+z}{t-z}\, ds,$$

$$\int_\Gamma c_1(t)\, ds = 0, \dots, \int_\Gamma c_1(t) \frac{ds}{t^{q-1}} = 0. \tag{12}$$

In view of the fact that the first q terms of the Maclaurin expansion of the function standing on the right-hand side of the equation must be equal to zero, the problem is reduced, by cancellation by z^q, to one of the cases considered above, plus the fulfillment of conditions (12).

Thus, problem (1) has been completely analyzed in the case when the Fourier series for $\gamma(z)$ has a finite number of terms with negative powers. Functions of this kind form an everywhere dense set in the space H_ν of functions satisfying a Hölder condition with exponent ν. But from the preceding considerations it is clear that, generally speaking, the solutions are unstable with respect to changes in $\gamma(z)$; therefore the indicated method can be used to solve problem (1) as an approximate method only in the case when it is known in advance that problem (1) is stable with respect to changes in $\gamma(z)$.

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