Mathematics

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Abstract

Full Text

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On K-Extendability of Polynomials

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In the present work we consider the problem of extendability of polynomials to the circle $|z| = 1$ of the plane of a complex variable, and give a new, very simple method for solving this problem.

The problem consists in adding to the terms of a given polynomial

$$f(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_{nz}^n$$

new summands in such a way that the roots of the polynomial

$$f_1(z) = f(z) + a_{n+1} z^{n+1} + \cdots + a_{mz}^m$$

lie on the circle $|z| = 1$.

From Newton' s formulas

$$\begin{aligned} s_1 + a_1 &= 0, \\ s_2 + a_1 s_1 + 2a_2 &= 0, \\ &\dots\dots\dots\dots \\ s_n + a_1 s_{n-1} + \cdots + na_n &= 0 \end{aligned}$$

we see that $s_1, s_2, \dots, s_n$ do not change under extension; moreover, if the roots of the polynomial $f_1(z)$ are denoted by $z_1, z_2, \dots, z_m$, then

$$s_k = \sum_{j=1}^{m} z_j^{-k}, \qquad k = 1, 2, \dots, n. \tag{1}$$

It is proved that, whatever $s_1, s_2, \dots, s_n$ may be, one can always choose $z_1, z_2, \dots, z_m$, equal to unity in modulus, so that equations (1) will be satisfied. Equations (1) admit solutions

$$s_k = \sum_{j=1}^{n} N_j(e^{ik\alpha_j} + e^{-ik\alpha_j})(\varepsilon_{1j}^k + \varepsilon_{2j}^k + \dots + \varepsilon_{jj}^k)e^{ik\psi_j}, \qquad k = 1, 2, \dots, n,$$

where $\varepsilon_{1j}, \varepsilon_{2j}, \dots, \varepsilon_{jj}$ are the roots of the equation

$$z^j = 1,$$

satisfying the conditions

$$\begin{aligned} \varepsilon_{1j} + \varepsilon_{2j} + \dots + \varepsilon_{jj} &= 0, \\ \varepsilon_{1j}^2 + \varepsilon_{2j}^2 + \dots + \varepsilon_{jj}^2 &= 0, \\ &\dots\dots\dots\dots \\ \varepsilon_{1j}^j + \varepsilon_{2j}^j + \dots + \varepsilon_{jj}^j &= j; \end{aligned}$$

N_j are sufficiently large positive integers denoting the multiplicities of the corresponding roots. Here α_j and ψ_j are found successively for $j = 1, 2, \dots, n$.

For $j = 1$, from the first equation we find

$$s_1 = N_1 \left(e^{i\alpha_1} + e^{-i\alpha_1}\right) e^{i\psi_1} = 2N_1 \cos\alpha_1 \, e^{i\psi_1}.$$

If $s_1 = \rho_1 e^{i\varphi_1}$, then to determine α_1 and ψ_1 we obtain two equations

$$\rho_1 = 2N_1 \cos\alpha_1, \qquad e^{i\varphi_1} = e^{i\psi_1},$$

which can be satisfied by setting

$$\cos\alpha_1 = \frac{\rho_1}{2N_1}, \qquad \psi_1 = \varphi_1,$$

which is possible for sufficiently large N_1. Having determined α_1 and ψ_1 from the first equation, we next find α_2 and ψ_2 from the second equation. Then we shall have

$$s_1 = 2N_1 e^{i\psi_1} \cos\alpha_1 + 2(\varepsilon_{12} + \varepsilon_{22}) N_2 e^{i\psi_2} \cos\alpha_2,$$

$$s_2 = 2N_1 e^{2i\psi_1} \cos 2\alpha_1 + 2(\varepsilon_{12}^2 + \varepsilon_{22}^2) N_2 e^{2i\psi_2} \cos 2\alpha_2.$$

Consequently, we shall have

$$s_2 - 2N_1 e^{2i\psi_1} \cos 2\alpha_1 = 4N_2 e^{2i\psi_2} \cos 2\alpha_2,$$

and, if

$$s_2 - 2N_1 e^{2i\psi_1} \cos 2\alpha_1 = \rho_2 e^{i\varphi_2},$$

then we take

$$2\psi_2 = \varphi_2, \qquad \cos 2\alpha_2 = \frac{\rho_2}{4N_2},$$

which is possible for sufficiently large N_2. In an analogous manner, from the k-th equation α_k and ψ_k will be found after $\alpha_1, \dots, \alpha_{k-1}, \psi_1, \dots, \psi_{k-1}$ have been determined from the preceding equations.

Further, without violating the generality of the theorem, one may remove from the circle $|z| = 1$ a subset of points of measure zero. In this case the following solutions of equations (1) are possible:

$$s_k = \sum_{j=1}^{n} N_j \left(e^{ik\beta_j} + e^{-ik\beta_j}\right)\left(e^{ik\alpha_j} + e^{-ik\alpha_j}\right)\left(\varepsilon_{1j}^k + \varepsilon_{2j}^k + \dots + \varepsilon_{jj}^k\right) e^{ik\psi_j},$$

$$k = 1, 2, \dots, n.$$

Using these formulas we successively compute $\alpha_1, \dots, \alpha_k, \psi_1, \dots, \psi_n$ first for $\beta_j = 0$; then we vary β_j on the segment $[0, \varepsilon]$, ensuring that the roots

$$\left(e^{i\beta_j} + e^{-i\beta_j}\right)\left(\varepsilon_{1j}^{-1} + \varepsilon_{2j}^{-1} + \dots + \varepsilon_{jj}^{-1}\right)\left(e^{i\alpha_j} + e^{-i\alpha_j}\right) e^{-i\psi_j}$$

fall into the remaining set. At the same time $d\alpha_j / d\beta_j = 0$ for $\beta_j = 0$. For example, for $j = 1$,

$$s_1 = 2N_1 e^{i\psi_1} \cos\alpha_1 \left(e^{i\beta_1} + e^{-i\beta_1}\right).$$

Then

$$\frac{s_1}{4N_1 \cos\beta_1} = e^{i\psi_1} \cos\alpha_1.$$

Putting $\beta_1 = 0$, we find α_1 and ψ_1 by the preceding method. Next we vary β_1 on the segment $[0, \varepsilon]$, ensuring that the roots

$$\left(e^{i\beta_1}+e^{-i\beta_1}\right)\left(e^{i\alpha_1}+e^{-i\alpha_1}\right)e^{-i\psi_1}$$

fall into the remaining set. Since $s_1=\rho_1 e^{i\varphi_1}$, we have $\cos\alpha_1=\rho_1/4N_1\cos\beta_1$. Differentiating, we obtain

$$-\sin\alpha_1\frac{d\alpha_1}{d\beta_1}=\frac{\rho_1\sin\beta_1}{4N_1\cos^2\beta_1},$$

and for $\beta_1=0$, $d\alpha_1/d\beta_1=0$, i.e. α_1 is approximately constant when β_1 varies on the segment $[0,\varepsilon]$ for sufficiently small ε.

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CITED LITERATURE

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2. P. S. Aleksandrov, A. N. Kolmogorov, *Introduction to the Theory of Functions of a Real Variable*, Moscow—Leningrad, 1938.

Note: Figure translations are in progress. See original paper for figures.