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Abstract

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SOME REMARKS ON THE RIGIDITY OF SURFACES OF POSITIVE CURVATURE WITH BOUNDARY

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Let a surface of positive Gaussian curvature K with boundary be given. Assuming that it is referred to an isometric system of coordinates, in which the linear element is represented in the form

$$ds^2 = \Lambda(x, y)(dx^2 + dy^2),$$

we write the Gauss–Peterson–Codazzi equations in the following form:

$$\frac{\partial \Lambda w}{\partial \bar{z}} + \frac{\partial \sqrt{K + |w|^2}}{\partial z} = 0, \tag{1}$$

where $w(z) = \frac{1}{2}(b_2^2 - b_1^1) + ib_1^2$; b_j^i are the mixed components of the tensor of the second quadratic form; $z = x + iy$. Suppose that the radius vector of the surface $r(x, y) \in D_{3,p}(\Gamma)$, $p > 2$ (^{1}), where Γ is the domain of definition of the metric of the surface. Differentiation in (1) is understood in the generalized Sobolev sense. The present note is devoted to the study of the behavior of the function $w(z)$ —a solution of equation (1)—under continuous bendings of the surface. Since this function, together with the metric, determines the form of the surface, the uniqueness of the solution of equation (1) for a given function $\Lambda(z)$ and under one or another set of conditions on the solution means the uniqueness, up to trivial transformations, of the determination of the surface by the metric and these conditions.

Let the surface (Λ, w_1) be subjected to a bending and be transformed thereby into the surface (Λ, w_2). Forming the equation satisfied by the difference of solutions of equation (1), $w^* = w_2 - w_1$, we see that it is a uniformly elliptic homogeneous equation of the form

$$\frac{\partial w^*}{\partial \bar{z}} + \mu_1(z)\frac{\partial w^*}{\partial z} + \mu_2(z)\frac{\partial \overline{w^*}}{\partial z} + a(z)w^* = 0, \tag{2}$$

where μ_1 and μ_2 are measurable functions in Γ; $a \in L_p(\Gamma)$, $p > 2$, if the functions w_1 and w_2 belong to $D_{1,p}(\Gamma)$, $p > 2$; $|\mu_1| + |\mu_2| \leq \mu_0 < 1$, μ_0 is a number. Consequently, for $w^*(z)$ there is a representation of the form

$$w^*(z) = \Phi(\zeta(z))e^{\varphi(z)},$$

where Φ is a function analytic in the domain $\zeta(\Gamma)$; $\zeta(z)$ is a certain homeomorphism of the z-plane onto the ζ-plane; $\varphi(z)$ is a function bounded in modulus.

It follows that

Lemma 1. *If a solution of equation (2) vanishes at the points of some set having a limit point inside* Γ*, then* $w^*(z) \equiv 0$ *everywhere.*

Suppose that Γ belongs to the class $C^{(1)}_{\alpha,\gamma_1\cdots\gamma_k}(l)$, $0 < \alpha \leqslant 1$, $\gamma_i > 0$. Then the following holds.

Lemma 2. *If* $w^* = 0$ *on some set of contour points of positive linear measure, then* $w^* = 0$ *everywhere.*

Let b be the normal curvature of the surface in some direction, and $b_\perp$ in the perpendicular direction. We shall call

$$E' = \frac{b - b_\perp}{2}$$

the Euler difference in the given direction, and $K' = bb_\perp$ the curvature in the given direction. If H is the mean curvature of the surface, then $E' = \sqrt{H^2 - K'}$. The Euler difference and the curvature in the direction of the lines of curvature coincide with the ordinary Euler difference and the Gaussian curvature. It is easy to obtain the formula (^{2})

$$Aw = E' = i\sqrt{\overline{K}' - K}, \tag{3}$$

where A is a coefficient depending on the direction.

Consider two isometric surfaces of positive Gaussian curvature F_1 and F_2. Suppose that on these surfaces, at corresponding points and in two fixed corresponding directions, the Euler differences and curvatures are equal. From formula (3) it then follows that $w_1 = w_2$ at corresponding points. From Lemmas 1 and 2 we obtain the following propositions.

Theorem 1. *If the sets of corresponding points of the surfaces* F_1 *and* F_2 *at which* $E_1' = E_2'$ *and* $K_1' = K_2'$ *have limit points on these surfaces, then* F_1 *and* F_2 *are congruent or symmetric.*

Theorem 2. *If* $E_1' = E_2'$ *and* $K_1' = K_2'$ *on sets of positive measure of corresponding points of the boundaries of the isometric surfaces* F_1 *and* F_2*, then the surfaces* F_1 *and* F_2 *are congruent or symmetric.*

From these theorems, in particular, there follows an analogue of Vekua's theorem, proved for infinitely small bendings: if an arbitrarily small arc is fixed on a surface of positive curvature, the surface admits no continuous bendings. (This arc may coincide with a part of the boundary of the surface, or may lie on the surface itself.) Indeed, in this case along the arc the spatial curvatures K_1 and K_2 of the arcs on F_1 and F_2 and the torsions $\varkappa_1$ and $\varkappa_2$ are equal. Consequently, along the arc $b_1 = b_2$. Hence it follows that $E_1' = E_2'$ and $K_1' = K_2'$ in the direction of the arc.

Let us also note that a surface with fixed boundary is unbendable. To fix the boundary completely means at the same time to fix on the boundary the functions E' and K' in the direction of the boundary. From the work (^{2}) it follows that, in order to achieve the same result, it is enough to fix on the boundary one of these functions. From Theorems 1 and 2 it follows that, in order to achieve the same goal, it is enough to fix E' and K' on an arbitrarily small arc of the boundary. Moreover, it follows from this that the theorems of the work (^{2}) are also true when the conditions occurring in them are prescribed not on the boundary, but on an arbitrary closed Jordan curve of class $C^1_{\alpha,\gamma_1\cdots\gamma_k}$ lying on the surface itself. In this case such a condition may be regarded as a boundary condition for the part of the surface lying inside this curve. From the unique determination of this part there then follows the unique determination of the whole surface.

Theorem 3. *Let a surface F_1 be given which contains a set of umbilic points having a limit point inside the surface. If the points of the surface F_2 corresponding under the isometry are umbilic, then F_1 and F_2 are congruent or symmetric.*

Theorem 4. *Let there lie on the boundary of the surface F_1 a set of umbilic points of positive linear measure. If the corresponding points of the isometric surface F_2 are umbilic, then the surfaces are congruent or symmetric.*

This means, for example, that when an arbitrarily small part of a piece of a spherical surface is bent, the bending propagates to the entire surface, and the Euler difference $\sqrt{H^2 - K}$ becomes everywhere nonzero, except perhaps on a set of points of the surface having no limit points on it, or on a set of boundary points of measure zero.

Theorem 4 may be regarded as a strengthening of Theorem 5 $^{(2)}$, where the unique determination of a surface of positive curvature whose boundary consists of umbilic points is asserted.

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Note: Figure translations are in progress. See original paper for figures.

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