# ON A THEOREM OF N. M. KOROBOV

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## Abstract

## Full Text

## MATHEMATICS

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# ON A THEOREM OF N. M. KOROBOV

*(Presented by Academician I. M. Vinogradov on 23 IV 1960)*

N. M. Korobov proved, ([3]), that if

$$\alpha_1, \alpha_2, \dots, \alpha_p, \dots \tag{1}$$

is a completely uniformly distributed sequence, then the sequence

$$[\alpha_1 q],\ [\alpha_2 q], \dots, [\alpha_p q], \dots$$

is a normal sequence of digits*.

In the present work we prove a theorem extending this theorem.

**Theorem.** The system of sequences

$$\begin{aligned} &\varepsilon_1,\ \varepsilon_2, \dots, \varepsilon_p, \dots, \\ &\delta_1,\ \delta_2, \dots, \delta_p, \dots, \end{aligned} \tag{2}$$

where

$$\varepsilon_j = [\alpha_j q_1], \qquad \delta_j = [q_2\{\alpha_j q_1\}],$$

is jointly normal.

**Proof.** Let $s = 1$. Take the one-column matrix

$$\begin{pmatrix} \varepsilon \\ \delta \end{pmatrix},$$

$0 \leqslant \varepsilon \leqslant q_1 - 1,\ 0 \leqslant \delta \leqslant q_2 - 1$. The occurrence of such a matrix in the system of sequences (2) in the $j$-th place is possible if and only if the number $\alpha_j$ falls into the half-interval

$$\left(\frac{\varepsilon q_2 + \delta}{q_1 q_2},\ \frac{\varepsilon q_2 + \delta + 1}{q_1 q_2}\right).$$

Observe that the length of this half-interval is

$$\frac{1}{q_1 q_2}.$$

Since the sequence (1) is completely uniformly distributed, the asymptotic frequency of occurrence of the matrix

$$\begin{pmatrix} \varepsilon \\ \delta \end{pmatrix}$$

in the system of sequences (2) is equal to

$$\frac{1}{q_1 q_2}.$$

Similarly, it is proved that for any natural number $s$ and any $s$-column matrix

$$\begin{pmatrix} \bar{\varepsilon}_1, & \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_s \\ \bar{\delta}_1, & \bar{\delta}_2, \dots, \bar{\delta}_s \end{pmatrix},$$

where $0 \leqslant \bar{\varepsilon}_k \leqslant q_1 - 1,\ 0 \leqslant \bar{\delta}_k \leqslant q_2 - 1,\ k = 1, 2, \dots, s$, the asymptotic frequency of occurrence in the system of sequences (2) is equal to

$$\frac{1}{q_2^s q_1^s}.$$

Consequently, the system of sequences (2) is jointly normal.

In the paper ([2])** the following problem was solved.

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* For the concepts of a normal sequence, jointly normal sequences, and a completely uniformly distributed sequence, see ([1]), §§ 1, 7, 12.

** For an exposition of this paper, see ([1]).

Let there be given a normal sequence consisting of the symbols $0, 1, \dots, q_1 - 1$,

$$\varepsilon_1,\ \varepsilon_2, \dots, \varepsilon_P, \dots \tag{3}$$

Construct a sequence, consisting of the symbols $0, 1, \dots, q_1 - 1$,

$$\delta_1,\ \delta_2, \dots, \delta_P, \dots \tag{4}$$

so that the system of sequences

$$\begin{matrix} \varepsilon_1,\ \varepsilon_2, \dots, \varepsilon_P, \dots, \\ \delta_1,\ \delta_2, \dots, \delta_P, \dots \end{matrix} \tag{5}$$

is jointly normal.

In § 7 of paper [1] a method is presented for constructing, for a given sequence, any number $l$ of sequences consisting of the symbols $0, 1, \dots, q_1 - 1$, such that the system of sequences

$$\varepsilon_1,\ \varepsilon_2, \dots, \varepsilon_P, \dots,$$

$$\varepsilon_1^{(1)},\ \varepsilon_2^{(1)}, \dots, \varepsilon_P^{(1)}, \dots,$$

$$\dots\dots\dots\dots\dots$$

$$\varepsilon_1^{(l)},\ \varepsilon_2^{(l)}, \dots, \varepsilon_P^{(l)}, \dots$$

is jointly normal.

With the aid of the theorem proved in the present paper, from the sequence (3) we shall construct a sequence consisting of the symbols $0, 1, \dots, q_2 - 1$,

$$\delta_1,\ \delta_2, \dots, \delta_P, \dots$$

such that the system of sequences will be jointly normal.

In paper [1], § 15, the construction is given of a completely uniformly distributed sequence

$$\alpha_1,\ \alpha_2, \dots, \alpha_P, \dots,$$

such that $\varepsilon_j = [q_1, \alpha_j]$. Taking $\delta_j = [q_2\{q_1\alpha_j\}]$, we obtain a sequence $\delta_1, \delta_2, \dots, \delta_P, \dots$ such that the system of sequences (5) is jointly normal.

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## CITED LITERATURE

[1] A. G. Postnikov, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, No. 57 (1960). [2] L. P. Starchenko, Izv. AN SSSR, Ser. Matem., **22**, 757 (1958), [3] N. M. Korobov, Izv. AN SSSR, Ser. Matem., **14**, 215 (1950).

*Note: Figure translations are in progress. See original paper for figures.*