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Abstract

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MATHEMATICS

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EXTREMAL ESTIMATES OF THE DERIVATIVE OF A TRIGONOMETRIC POLYNOMIAL ON AN INTERVAL SHORTER THAN THE PERIOD

(Presented by Academician S. N. Bernstein on 29 VIII 1959)

Set

$$t_n(\theta) = -\cos 2n \arccos \frac{\sin(\theta/2)}{\sin(\omega/2)}, \quad u_n(\theta) = \sin 2n \arccos \frac{\sin(\theta/2)}{\sin(\omega/2)};$$

it is easy to verify that $t_n(\theta)$ is a trigonometric polynomial of order n for $-\omega \leq \theta \leq \omega$, $0 < \omega < \pi$.

Theorem. If a trigonometric polynomial $s_n(\theta)$ of order n satisfies the inequality

$$|s_n(\theta)| \leq 1, \quad -\omega \leq \theta \leq \omega, \quad 0 < \omega < \pi, \quad (1)$$

then

$$|s'_n(\theta)| \leq |t'_n(\theta) + iu'_n(\theta)| = n \cos(\theta/2) [\sin^2(\omega/2) - \sin^2(\theta/2)]^{-1/2},$$

$$-\omega < \theta < \omega, \quad (2)$$

and for $n > [3 \operatorname{tg}^2(\omega/2) + 1]^{1/2}/2$

$$|s'_n(\theta)| \leq t'_n(\omega) = 2n^2 \operatorname{ctg}(\omega/2), \quad -\omega \leq \theta \leq \omega. \quad (3)$$

In (2) equality is attained only for polynomials $s_n(\theta) = \gamma t_n(\theta)$, $|\gamma| = 1$, at the $2n$ points θ_ν that are zeros of $t_n(\theta)$ on $[-\omega, \omega]$; in (3), for the same polynomials, but only at the points $\theta = \pm\omega$.

Inequalities (2) and (3) are analogous to the classical inequalities of S. N. Bernstein and A. A. Markov for algebraic polynomials. Inequality (2) refines a known result of I. I. Privalov ⁽¹⁾, who proved that (1) implies the inequality

$|s'_n(\theta)| \leq C_{\omega, \varepsilon} n$ for $-\omega + \varepsilon \leq \theta \leq \omega - \varepsilon$, $0 < \varepsilon < \omega/2$, where $C_{\omega, \varepsilon}$ is a constant depending on ω and ε . Inequality (3) correspondingly refines Jackson's estimate (2): $|s'_n(\theta)| \leq C_\omega n^2$, $-\omega \leq \theta \leq \omega$, where C_ω is a constant depending on ω .

After inequality (2) had been established, it could have been derived from a general theorem of N. I. Akhiezer and B. Ya. Levin (3) by investigating the corresponding conformal mapping; but since that theorem itself is proved rather intricately, another path is chosen here, connected with the author's previous works and admitting further generalizations. Note that if $|s_n(\theta)| \leq 1$ for $-\pi \leq \theta \leq \pi$, then passage to the limit $\omega \rightarrow \pi$ in the right-hand side of (2) gives the classical S. N. Bernstein inequality $|s'_n(\theta)| \leq n$, $-\pi \leq \theta \leq \pi$. In all subsequent arguments we shall assume the coefficients of the polynomial $s_n(\theta)$ to be real, since passing to complex coefficients presents no difficulty (see (4), p. 45).

Lemma. If $s_n(\theta)$ is a trigonometric polynomial satisfying the inequality $|s_n(\theta)| < 1$, $-\omega \leq \theta \leq \omega$, then for all real β the function

$$F(\theta; \beta) = \cos \beta t_n(\theta) + \sin \beta u_n(\theta) - s_n(\theta)$$

has only simple zeros on $[-\omega, \omega]$.

Make the substitution $x \operatorname{tg}(\omega/2) = \operatorname{tg}(\theta/2)$ and, in addition, put $a = \operatorname{tg}(\omega/2)$, $\alpha = \sqrt{1 + a^2}$. Then

$$s_n(\theta) = (1 + a^2 x^2)^{-n} P_{2n}(x),$$

$$F(\theta; \beta) = (1 + a^2 x^2)^{-n} [\cos \beta M_{2n}(x) + \sin \beta \sqrt{1 - x^2} N_{2n-1}(x) - P_{2n}(x)],$$

where $P_{2n}(x)$ is an algebraic polynomial of degree $2n$, while $M_{2n}(x)$ and $N_{2n-1}(x)$ are algebraic polynomials of degrees $2n$ and $2n - 1$, respectively, which are determined by the formulas

$$M_{2n}(x) = \Re (\alpha x + i\sqrt{1 - x^2})^{2n}, \quad \sqrt{1 - x^2} N_{2n-1}(x) = \Im (\alpha x + i\sqrt{1 - x^2})^{2n}.$$

All the zeros of $M_{2n}(x)$ and $N_{2n-1}(x)$ lie in the interval $(-1, 1)$ and mutually interlace (see, for example, (5)). From (1) there follows the inequality

$$|P_{2n}(x)| \leq (1 + a^2 x^2)^n = |M_{2n}(x) + i\sqrt{1 - x^2} N_{2n-1}(x)|, \quad -1 \leq x \leq 1. \quad (4)$$

But, as can be shown by following the method of my note (6), it follows from (4) that $\cos \beta M_{2n}(x) + \sin \beta \sqrt{1 - x^2} N_{2n-1}(x) - P_{2n}(x)$ cannot have multiple zeros

on the interval $-1 \leq x \leq 1$. Hence we conclude that the function $F(\theta; \beta)$ has only simple zeros on the interval $-\omega \leq \theta \leq \omega$, since $dx/d\theta = (1 + a^2x^2)/2a$ does not vanish. The lemma is proved.

Suppose that at some point θ_0 , $-\omega < \theta_0 < \omega$, inequality (2) is violated, i.e.

$$s'_n(\theta_0) \geq |t'_n(\theta_0) + iu'_n(\theta_0)| \quad (s_n(\theta) \neq \pm t_n(\theta)). \quad (5)$$

Choose the parameter β occurring in the function $F(\theta; \beta)$ according to the condition

$$[\cos \beta t_n(\theta_0) + \sin \beta u_n(\theta_0)]s'_n(\theta_0) - [\cos \beta t'_n(\theta_0) + \sin \beta u'_n(\theta_0)]s_n(\theta_0) = 0, \quad (6)$$

which, of course, is always possible. Next choose the parameter λ according to the condition

$$\cos \beta t'_n(\theta_0) + \sin \beta u'_n(\theta_0) - \lambda s'_n(\theta_0) = 0. \quad (7)$$

From (6) and (7) it follows that

$$\cos \beta t_n(\theta_0) + \sin \beta u_n(\theta_0) - \lambda s_n(\theta_0) = 0. \quad (8)$$

Moreover, from (7) and (5) it follows that $-1 \leq \lambda \leq 1$, since by the Cauchy-Bunyakovsky inequality we have

$$|\lambda s'_n(\theta_0)| \leq [t'^2_n(\theta_0) + u'^2_n(\theta_0)]^{1/2}.$$

Consequently, the polynomial $\lambda s_n(\theta)$ satisfies the inequality $|\lambda s_n(\theta)| < 1$, and the corresponding function $F(\theta; \beta)$ has, at the point θ_0 , in view of (7) and (8), a double zero, which contradicts the lemma. Inequality (2) is established.

We pass to the proof of inequality (3). Let $\xi_1 < \xi_2 < \dots < \xi_{2n-1}$ be the zeros of the polynomial $t'_n(\theta)$ on $[-\omega, \omega]$. First of all, let us show that on $[\xi_{2n-1}, \omega]$ the polynomial $t'_n(\theta)$ increases monotonically. In view of the oddness of the polynomial $t'_n(\theta)$, it has a zero at the point $\theta = \pi$. By Rolle's theorem, in the interval $\xi_{2n-1} < \theta < \pi$ there lies one and only one zero of its derivative $t''_n(\theta)$. It is easy to verify that this zero is located to the right of the point ω . Indeed, since at the point ξ_{2n-1} the polynomial $t_{2n}(\theta)$ has a minimum, $t''_n(\xi_{2n-1}) > 0$, while on the other hand a direct calculation gives $t''_n(\omega) > 0$ for $2n > [3 \operatorname{tg}^2(\omega/2) + 1]^{1/2}$. Consequently, $t''_n(\theta) > 0$ on $[\xi_{2n-1}, \omega]$, and $t'_n(\theta)$ increases monotonically.

is attained on this interval. From this we shall derive that $|t'_n(\theta)| \leq t'(\omega) = 2n^2 \operatorname{ctg}(\omega/2)$ for $-\omega \leq \theta \leq \omega$. Indeed, this follows directly for the intervals $-\omega \leq \theta \leq \xi_1$ and $\xi_{2n-1} \leq \theta \leq \omega$ from the monotonicity of $t'_n(\theta)$ on these

intervals. If $0 \leq \theta \leq \theta_{2n}$, θ_{2n} being the rightmost zero of $t_n(\theta)$ on $[-\omega, \omega]$, then, observing that the zeros of $t_n(\theta)$ coincide with the zeros of $u'_n(\theta)$ and that $|t'_n(\theta) + iu'_n(\theta)|$ increases monotonically on $[0, \theta_{2n}]$, from (2) we conclude that

$$|t'_n(\theta)| \leq |t'_n(\theta_{2n}) + iu'_n(\theta_{2n})| = t'(\theta_{2n}),$$

but $t'(\theta_{2n}) \leq t'_n(\omega)$, since $\xi_{2n-1} < \theta_{2n} < \omega$.

The proof of inequality (3) is equivalent to proving that, among the trigonometric polynomials $s_n(\theta)$ of order n with the given value

$$\max |s'_n(\theta)| = 2n^2 \operatorname{ctg}(\omega/2)$$

on the interval $-\omega \leq \theta \leq \omega$, the polynomial least deviating from zero on this interval is the polynomial $t_n(\theta)$ (for $2n > [3 \operatorname{tg}^2(\omega/2) + 1]^{1/2}$). Suppose that this is not so and that the extremal polynomial is $\sigma_n(\theta) \neq \pm t_n(\theta)$. Clearly, $|\sigma_n(\theta)| < 1$ on $[-\omega, \omega]$, hence

$$|\sigma'_n(\theta)| < |t'_n(\theta) + iu'_n(\theta)|.$$

If $\theta_1 \leq \theta \leq \theta_{2n}$ (θ_1, θ_{2n} are the extreme zeros of $t_n(\theta)$ on $[-\omega, \omega]$), then

$$|\sigma'_n(\theta)| < |t'_n(\theta) + iu'_n(\theta)| \leq |t'_n(\theta_{2n}) + iu'_n(\theta_{2n})| = t'_n(\theta_{2n}).$$

Therefore the point θ_0 , at which $\max |\sigma'_n(\theta)| = |\sigma'_n(\theta_0)|$ is attained, must lie in one of the intervals $(-\omega, \theta_1)$, (θ_{2n}, ω) . Suppose, for definiteness, that $\theta_{2n} < \theta_0 < \omega$. The polynomial $\sigma_n(\theta)$ must have on $[-\omega, \omega]$ at least $2n$ points of maximum deviation with successively opposite signs. Suppose this is not so and that the number of the indicated points of maximum deviation is $2n - 1$; denote them by $\varphi_1, \dots, \varphi_{2n-1}$. Then one can construct such a trigonometric polynomial $S_n(\theta)$ of order $\leq n$ which, at the points φ_ν , would have successively opposite signs and for which $S'_n(\theta_0) = 0$. Put

$$R_{2n}(x) = \sum_{\nu=1}^{2n-1} \frac{(-1)^\nu Q(x)}{(x - x_\nu) Q'(x_\nu)} + C(x - \mu)Q(x), \quad Q(x) = \prod_{\nu=1}^{2n-1} (x - x_\nu),$$

where $ax_\nu = \operatorname{tg}(\varphi_\nu/2)$, $a = \operatorname{tg}(\omega/2)$, and C and μ are certain constants, which we shall now choose in a suitable way. Since $Q(x)$ has no multiple zeros, the parameter μ can be chosen so that

$$(1 + a^2 x_0^2)[(x_0 - \mu)Q'(x_0) + Q(x_0)] - 2na^2 x_0(x_0 - \mu)Q(x_0) \neq 0, \quad (9)$$

where $ax_0 = \operatorname{tg}(\theta_0/2)$. Now put

$$S_n(\theta) = (1 + a^2 x^2)^{-n} R_{2n}(x), \quad ax = \operatorname{tg}(\theta/2),$$

then

$$2aS'_n(\theta) = (1 + a^2 x^2)^{-n} \{ (1 + a^2 x^2) R'_{2n}(x) - 2na^2 x R_{2n}(x) \},$$

and, thanks to (9), we can choose the parameter C so that $S'_n(\theta_0) = 0$. On the other hand, $S_n(\varphi_\nu)S_n(\varphi_{\nu+1}) < 0$. Consequently, there is such an ε that the polynomial $\sigma_n(\theta) + \varepsilon S_n(\theta)$ will deviate from zero on $[-\omega, \omega]$ less than $\sigma_n(\theta)$.

The polynomial $\sigma_n(\theta)$ cannot have in the open interval $(-\omega, \omega)$ $2n$ points of maximum deviation with successively opposite signs, since then we would have

$$\sigma_n(\theta) = 2n \operatorname{ctg}(\omega/2) \cos(n\theta - \beta),$$

$$\max |\sigma_n(\theta)| = 2n \operatorname{ctg}(\omega/2) > 1 = \max |t_n(\theta)|.$$

The set of points of maximum deviation of the polynomial $\sigma_n(\theta)$ also cannot consist of $2n - 1$ points lying inside the interval $(-\omega, \omega)$ and one point coinciding with one of the endpoints $\pm\omega$, since then the po

the polynomial $\sigma_n(\theta + \delta)$, for sufficiently small δ , would have only $2n - 1$ points of maximum deviation on $[-\omega, \omega]$, and moreover

$$\max |\sigma_n(\theta)| = \max |\sigma_n(\theta + \delta)|.$$

But the polynomial $\sigma_n(\theta + \delta)$, by the preceding, cannot be the least deviating from zero.

Finally, suppose that the polynomial $\sigma_n(\theta)$ has $2n - 2$ points of maximum deviation inside $(-\omega, \omega)$, and that both endpoints $\pm\omega$ are such points. Let

$$\max_{-\omega \leq \theta \leq \omega} |\sigma_n(\theta)| = L < 1$$

and

$$\sigma_n(\theta) = (1 + a^2 x^2)^{-n} \Phi_{2n}(x), \quad ax = \operatorname{tg}(\theta/2), \quad (10)$$

where $\Phi_{2n}(x)$ is the corresponding algebraic polynomial of degree $2n$, satisfying the inequality (see (4))

$$|\Phi_{2n}(x)| \leq L |M_{2n}(x) + i\sqrt{1-x^2} N_{2n-1}(x)|, \quad -1 \leq x \leq 1. \quad (11)$$

From (11) follows (7), that $|\Phi'_{2n}(1)| < LM'_{2n}(1)$. From (10) we have $\Phi_{2n}(1) = LM_{2n}(1)$. Direct calculation gives

$$\begin{aligned} t'_n(\omega) - \sigma'_n(\omega) &= \\ &= \frac{1}{2a} (1 + a^2)^{-n} \{ (1 + a^2) [M'_{2n}(1) - \Phi'_{2n}(1)] - 2na^2 [M_{2n}(1) - \Phi_{2n}(1)] \} = \\ &= \frac{1}{2a} (1 + a^2)^{-n} \{ (1 + a^2) [LM'_{2n}(1) - \Phi'_{2n}(1)] + \\ &\quad + (1 - L) [(1 + a^2) M'_{2n}(1) - 2na^2 M_{2n}(1)] \} = \\ &= \frac{1}{2a} (1 + a^2)^{-n+1} [LM'_{2n}(1) - \Phi'_{2n}(1)] + (1 - L) 2n^2 \operatorname{ctg}(\omega/2) > 0. \end{aligned}$$

In addition, by virtue of our assumptions,

$$t'_n(\theta_{2n}) - \sigma'_n(\theta_{2n}) > 0, \quad t'_n(\theta_0) - \sigma'_n(\theta_0) < 0.$$

Consequently, the polynomial $t'_n(\theta) - \sigma'_n(\theta)$ of order n has at least 2 zeros in the interval (θ_{2n}, ω) , and, by virtue of the inequalities

$$|\sigma'_n(\theta_\nu)| < |t'_n(\theta_\nu) + iu'_n(\theta_\nu)| = |t'_n(\theta_\nu)|,$$

at least $2n - 1$ zeros in the interval (θ_1, θ_{2n}) , i.e. $\geq 2n + 1$ zeros in the interval (θ_1, ω) of length $< 2\pi$, which is impossible. Thus, the extremal polynomial is the polynomial having $2n + 1$ points of maximum deviation with successively alternating signs, i.e. the polynomial $\pm t_n(\theta)$. The theorem is completely proved.

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Note: Figure translations are in progress. See original paper for figures.

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