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Abstract

Full Text

MATHEMATICS

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ON THE DIRICHLET PROBLEM FOR DOMAINS WITH ANGLES

(Presented by Academician I. M. Vinogradov, 10 XI 1959)

This note is devoted to the study of the differential properties of a function $U(x, y)$, harmonic in a plane domain Ω , in dependence on the differential properties of the boundary function $f(s)$ in the case when the boundary Γ of the domain Ω is piecewise smooth. The differential properties of U and f are determined in terms of membership of these functions, respectively, in the classes $W_p^r H^\alpha(\Omega)$ and $H_p(\Gamma)$ ($1 \leq p \leq \infty$), whose definition is given in the papers of S. M. Nikol'skii ⁽¹⁾.

If the contour Γ is sufficiently smooth, then the following assertion holds, proved in the papers of N. P. Mozzhezerova ⁽²⁾, S. M. Nikol'skii ⁽³⁾, and Ya. S. Bugrov ⁽⁴⁾; it represents a strengthening (for $p = \infty$) and an extension to the case of finite p of the corresponding results of Lichtenstein ⁽⁵⁾, N. M. Günter ⁽⁶⁾, Kh. L. Smolinskii, and others.

If the function $f \in H_p^r(\Gamma)$, then for the harmonic function U , for which $U|_\Gamma = f$, one has $U \in W_p^\rho H^\alpha(\Omega)$, $\rho + \alpha = r + \frac{1}{p}$. In the case of piecewise smooth contours Γ , this property is far from always preserved.

In the paper of S. M. Nikol'skii ⁽³⁾, necessary and sufficient conditions are given which the boundary function f , prescribed on the boundary Γ of the square Ω , must satisfy in order that the corresponding function harmonic in Ω belong to $W_p^r H^\alpha(\Omega)$. Uolsh and Yang considered a similar problem for the square for $p = \infty$ and $r = 0$ ⁽⁷⁾.

These investigations are extended in the present note to the case of arbitrary domains with nondegenerate angles, and arbitrary p and r . It turns out that the angles π/j , $j = 1, 2, \dots$ (and thus also the straight angle), are in a certain sense exceptional.

Let the domain Ω have a boundary Γ which is a piece of a smooth curve of class $H_\infty^{r+2+1/p}$, whose ends meet at an angle ω , and suppose that some finite parts of Γ adjoining the vertex of the angle P_0 are straight-line segments. We shall assume that the value of the arc $s = 0$ corresponds to the vertex of the angle and that its length is equal to l . The boundary function f will be given as a function $f(s)$ of the arc s on the interval $0 \leq s \leq l$.

Theorem 1. Let $r, r-1/p$ be nonintegral positive numbers and let the harmonic function $U \in W_p^r H^\alpha(\Omega)$, $\bar{r} + \alpha = r$.

Then the function $f(s) = U|_\Gamma$ satisfies the following conditions:

- 1) $f(s) \in H_p^{r-1/p}([0, l])$;
- 2) in the case $\omega = \pi/j$, $j = 1, 2, \dots$,

$$f^{(kj)}(+0) = (-1)^k f^{(kj)}(-0)$$

for all $k = 0, 1, 2, \dots$, for which $kj < \rho$, where $\rho = r - 1/p = \bar{\rho} + \beta$, $\bar{\rho}$ is an integer, $0 < \beta < 1$.

If $\bar{\rho} = mj$ (m an integer), then, in addition, the inequality

$$\left(\int_0^h |f^{(mj)}(u) - (-1)^m f^{(mj)}(-u)|^p du \right)^{1/p} \leq ch^\beta$$

holds.

Remark. Property 1) follows from the general embedding theorem of S. M. Nikol'skii (1³); property 2) reflects the harmonicity of the function U in a neighborhood of the point P_0 . It turns out that Theorem 1 is completely invertible only in the case when $\omega = \pi/j$, $j = 1, 2, \dots$, as is seen from the following theorem.

Theorem 2. Let $\omega = \pi/j$, $j = 1, 2, \dots$; r and $r - 1/p$ be noninteger positive numbers; $r - 2/p > 0$. Let $f(s)$ satisfy the conditions:

- 1) $f(s) \in H_p^{r-1/p}(M, [0, l])$;
- 2) $f^{(kj)}(+0) = (-1)^k f^{(kj)}(-0)$

for all $k = 0, 1, \dots$ for which $kj < \bar{\rho}$, where $\rho = r - 1/p = \bar{\rho} + \beta$, $\bar{\rho}$ is an integer, $0 < \beta < 1$, and in the case $\bar{\rho} = mj$ (m an integer) the inequality

$$\left(\int_0^h |f^{mj}(u) - (-1)^m f^{mj}(-u)|^p du \right)^{1/p} \leq c|h|^\beta$$

also holds.

Then the function $U(x, y)$, harmonic in Ω , with boundary function $f(s)$, belongs to the class $W_p^r H^\alpha(M_1, \Omega)$, where $r + \alpha = r$, and

$$M_1 \leq c_1 (M + c + \|f(s)\|_{L(p)}).$$

For $\omega = \pi/2$, Theorems 1 and 2 were proved by S. M. Nikol'skii (3).

Theorem 3. If ω is not equal to one of the numbers π/j , $j = 1, 2, \dots$, then the converse of Theorem 2 holds only for $r - 2/p < \pi/\omega$. In the case $r - 2/p > \pi/\omega$, one can construct a boundary function $f(s)$, having arbitrarily many derivatives for $0 \leq s \leq l$ and equal to zero on the segments $0 \leq s \leq \delta$, $l - \delta \leq s \leq l$, with the property that the harmonic function U , for which $U|_{\Gamma} = f$, does not belong to the class $W_p^r H^\alpha(\Omega)$, $r + \alpha = r$.

The proofs of the theorems were carried out by means of methods of the theory of double-layer potentials, as well as methods of the theory of conformal mappings. In the course of the proofs we needed the following lemmas.

Lemma 1. Let the contour Γ , bounding a two-dimensional domain Ω , be smooth everywhere, and let

$$\Gamma \in H_\infty^{r+2+1/p}(M_1), \quad r = r + \alpha.$$

If $\mu(s)$ is the density of a double-layer potential and $\mu(s) \in H_p^r(M, \Gamma)$, then the corresponding value of the double-layer potential on the contour Γ , $\bar{\mu}(s)$, belongs to the class $H_p^{r+1}(\bar{M}, \Gamma)$, where

$$\bar{M} < c\{M + \|\mu\|_{L_p}(\Gamma)\}.$$

Lemma 2. Let the contour Γ , bounding a two-dimensional domain Ω , be smooth everywhere, and let

$$\Gamma \in H_\infty^{r+2+1/p}(M_1).$$

If $\mu(s)$ is the density of the double-layer potential and $\mu(s) \in H_p^r(M, \Gamma)$, then $U(x, y)$, the double-layer potential with density $\mu(s)$, belongs to the class $W_p^r H^\alpha(\bar{M}, \Omega)$, where $r + 1/p = \rho = \bar{\rho} + \alpha$,

$$\bar{M} \leq c\{M + \|\mu\|_{L_p}\}.$$

Lemmas 1 and 2 in the three-dimensional case were proved by N. P. Mozherova (2); the proof for two-dimensional domains is analogous.

Lemma 3. Let G be a two-dimensional domain bounded by a piecewise smooth contour Γ with angles ω_l , $l = 0, 1, \dots, N$, at the points P_l , where $0 < \omega_l < 2\pi$; let each smooth piece

$$\Gamma \in H_\infty^{r+2+1/p}([s_k, s_{k+1}]),$$

where s_0, s_1, \dots, s_N are the values of the arc s corresponding to the corner points of Γ . If

$$f(s) \in H_p^r(M, [s_k, s_{k+1}]), \quad k = 0, 1, \dots, N - 1,$$

then for any domain $\Omega' \in \Omega$ such that

$$\rho(\Omega', P_k) > \varepsilon > 0, \quad k = 0, 1, \dots, N,$$

$$U \in W_p^{\bar{\rho}} H^\alpha(\bar{M}, \Omega'), \quad \rho = \bar{\rho} + \alpha = r + \frac{1}{p},$$

where

$$\bar{M} \leq c(\varepsilon)\{M + \|f\|_{L_p}\}.$$

Remark. Theorems 1, 2, and 3 are formulated in the corresponding way (and, with the aid of Lemma 3, proved) in the case of domains with an infinite number of corner points.

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Note: Figure translations are in progress. See original paper for figures.

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