ON THE SPACE OF BOUNDED REGULAR FUNCTIONS

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Abstract

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MATHEMATICS

V. P. KHAVIN

ON THE SPACE OF BOUNDED REGULAR FUNCTIONS

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1. Let R be the extended complex plane; F a closed set of its points; $G = R \setminus F \neq \Lambda$; $A(G)$ the totality of all functions regular in G; $A(F)$ the totality of all functions regular on F (two elements $\varphi_1, \varphi_2 \in A(F)$ such that $\varphi_1(z) = \varphi_2(z)$ for all z in some neighborhood of F are identified).

We shall consider only those regular functions which are equal to zero at $z = \infty$, if this point belongs to their set of definition.

An open set $g \supset F$, whose boundary ∂g consists of a finite number of closed rectifiable pairwise nonintersecting Jordan curves, will be called a canonical neighborhood of F. We introduce in $A(F)$ the natural algebraic operations and topology ([1,2]). Then every element $\psi \in A(G)$ can be identified with a certain additive functional Φ_ψ, defined in $A(F)$ by formula (1):

$$\Phi_\psi(x) = \frac{1}{2\pi i} \int_{\partial g_x} x(\zeta)\psi(\zeta)\, d\zeta. \tag{1}$$

Here ∂g_x is the boundary of a canonical neighborhood g_x of the set F, in the closure of which the function x is regular.

Many classes of functions regular in G can be completely characterized as additive functionals, defined on $A(F)$ and continuous in some topology weaker than the topology of $A(F)$ ([3]). In this note we shall indicate such a characterization of the class $B(G)$ of all functions regular and bounded in G.

2. Let μ be a complex measure defined on the Borel subsets of G; if there exists a set P, $\overline{P} = P \subset G$, such that for every Borel $e \subset F \setminus P$ $\mu(e) = 0$, then we shall say that μ is concentrated on P and that P is a support of μ. To every element $\varphi \in A(F)$ we assign the totality $M(\varphi)$ of all such measures μ with supports in G for which

$$\varphi(z) = \int_G \frac{d\mu_\zeta}{\zeta - z}$$

in some neighborhood of F. We now define on $A(F)$ a seminorm $|||\cdot|||$, putting

$$|||\varphi||| = \inf_{\mu\in M(\varphi)} \left\{ \int_G |d\mu| \right\}. \tag{2}$$

The linear set $A(F)$, endowed with the seminorm (2), is transformed in the usual way into a linear topological space (generally speaking, non-Hausdorff), which we shall denote by $\mathcal{L}(F)$. The space $\mathcal{L}(F)$ is certainly not complete.

Theorem 1. Let $f \in B(G)$; then the functional Φ_f is continuous in $\mathcal{L}(F)$, and

$$\|\Phi_f\| = \sup_{z\in G} |f(z)|.$$

If Φ is an additive functional continuous in $\mathcal{L}(F)$, then there exists $f \in B(G)$ such that $\Phi = \Phi_f$. This function f is unique.

Let the norm $\|f\|_{B(G)} = \max_{z\in G} |f(z)|$ be introduced in $B(G)$. Thus the normed space $B(G)$ turns out to be conjugate to $\mathcal{L}(F)$.

Corollary 1. Let $S_1(G)$ be the unit ball in $B(G)$, $\varphi \in A(F)$. Then

$$\sup_{f\in S_1(R)} \frac{1}{2\pi} \left| \int_{\partial g_\varphi} \varphi(\zeta) f(\zeta)\, d\zeta \right| = \inf_{\mu\in M(\varphi)} \int_G |d\mu|.$$

Here g_φ is a canonical neighborhood of F, in whose closure φ is regular. The supremum on the left is always attained.

This corollary is close to the well-known "duality relations" of S. Ya. Khavinson ([4]).

Corollary 2. Let G be a domain, $\infty \in G$. In order that $B(G)$ contain a function f different from identically zero, it is necessary and sufficient that for some $z_0 \in G$

$$|||\psi_{z_0}||| > 0 \qquad \left(\psi_{z_0}(z) = \frac{1}{z - z_0} \right).$$

Remark. Let $\mathcal{L}_1(F)$ be the totality of all measures with supports lying in G. For $\mu \in \mathcal{L}_1(F)$ put

$$\|\mu\|^1 = \int_G |d\mu|.$$

Let $\mathfrak{K}(F)$ be the subset of $\mathcal{L}_1(F)$ consisting of all measures μ such that

$$\int_G \frac{d\mu_\zeta}{\zeta - z} \equiv 0$$

in some neighborhood of F. Then the quotient space $\mathcal{L}_1(F)/\mathfrak{K}(F)$ is isomorphic to $\mathcal{L}(F)$. In some simple cases one can construct a complete space whose conjugate is isomorphic to $B(G)$. Thus, for example, if G is the unit disk, then such a space will be L/H_1. Here L is the space of all functions defined on the unit circle and summable (with the usual norm). For the definition of the set H_1, see ([5]). Hence it is not difficult to derive the following fact.

Let $a_1, a_2, \dots, a_n, \dots$ be a sequence of complex numbers,

$$|a_1| \le |a_2| \le \cdots \le |a_n| \le \cdots < 1.$$

The following assertions are equivalent:

$$1)\ \sum_1^\infty (1 - |a_n|) = \infty;$$

2) whatever $\varphi \in L$ and $\varepsilon > 0$ may be, there is a fraction

$$\sum \frac{\lambda_k}{z - a_k}$$

and $\psi \in H_1$ such that

$$\int_0^{2\pi} \left| \varphi(e^{i\theta}) - \psi(e^{i\theta}) - \sum \frac{\lambda_k}{e^{i\theta} - a_k} \right| d\theta < \varepsilon.$$

Here $\lambda_1, \lambda_2, \dots$ is a sequence of complex numbers, only finitely many of which are different from zero.

3. Let G be a domain, $\infty \in G$. The number

$$\Omega(F) = \sup_{f \in S_1(G)} \left| \int_{\partial g} f(\zeta)\, d\zeta \right|$$

is called the **analytic capacity** of F. The importance of this concept in approximation theory was discovered by A. G. Vitushkin (see, for example, ([6])).

Theorem 2.

$$\Omega(F) = \inf_{\mu \in M(1)} \int_G |d\mu|.$$

Here $1(z) \equiv 1$.

We shall say that F has finite enclosure if there exists a number $l(F)$ such that every neighborhood of F contains a canonical neighborhood whose boundary length does not exceed $l(F)$.

If F has finite enclosure, then any $f \in B(G)$ has the form

$$f(z) = \int_F \frac{d\nu_t}{t - z} \quad (z \in G),$$

where ν is a measure defined on the Borel subsets of F.

The proof follows easily from the theorems of our paper ([7]).

It follows from this remark that the analytic capacity is the exact upper bound of the "masses" $\left|\int_F d\nu\right|$ that can be placed on F so that the "potential" $\int_F \frac{d\nu}{t-z}$ at any point $z \in G$ does not exceed one in modulus.

Theorem 2 shows that, on the other hand, $\Omega(F)$ is the exact lower bound of the "masses" that can be placed outside F so that the "potential" they create in a neighborhood of F is equal to one. In particular, $\Omega(F) = 0$ if and only if an arbitrarily small "mass" situated outside F is capable of producing on F a uniformly large "potential."

Theorem 3. If $\Omega(F) = 0$, then, whatever $\varphi \in A(F)$ and $\varepsilon > 0$ may be, there exists a measure $\mu \in \mathcal{L}_1(F)$ such that

$$\varphi(z) = \int_G \frac{d\mu}{\xi - z}$$

for all z in some neighborhood of F, and

$$\int_G |d\mu| < \varepsilon.$$

Theorem 4. 1) If $\Omega(F) = 0$, then, whatever the continuous function λ on F and the number $\varepsilon > 0$ may be, there exists a fraction

$$R(z) = \sum_{k=1}^{n} \frac{\lambda_k}{z - a_k}$$

$(a_1, a_2, \dots, a_n \in G)$ such that

$$\max_{z \in F} |R(z) - \lambda(z)| < \varepsilon, \qquad \sum_{k=1}^{n} |\lambda_k| < \varepsilon.$$

2) If F has a finite cover and if, for every $\varepsilon > 0$, one can find a fraction

$$R(z) = \sum_{k=1}^{n} \frac{\lambda_k}{z - a_k} \quad (a_1, a_2, \dots, a_n \in G)$$

such that

$$\max_{z\in F}|1-R(z)|<\varepsilon,\qquad \sum_{k=1}^{n}|\lambda_k|<\varepsilon,$$

then $\Omega(F)=0$.*

4. **Theorem 5.** Let F contain a nondegenerate continuum or, being a discontinuum, have a finite cover and positive analytic capacity. Then for every $\varphi\in A(F)$, $\varphi\neq 0$, one can find an element $f\in B(G)$ such that $\Phi_f(\varphi)\neq 0$.

Corollary 1. Let F have a finite cover. Then the following alternative is valid: either $B(G)$ contains no elements different from zero, or $B(G)$ is dense in $A(G)$ (in the sense of uniform convergence on compact sets lying in G).

Corollary 2. Under the conditions of Corollary 1, either (2) is a norm on $A(F)$, or $\|\varphi\|=0$ for every $\varphi\in A(F)$.

Corollary 3. Under the conditions of Corollary 1, either $\mathcal{K}(F)$ is dense in $\mathcal{L}_1(F)$, or $\mathcal{K}(F)$ is closed in $\mathcal{L}_1(F)$.

The proof of Theorem 3 is based on the results of §§ 2 and 3 and on the following obvious lemma. Let F be a bounded discontinuum; let λ be a function regular on F, $\lambda\neq 0$, and vanishing on F only at the points $a_1,a_2,\dots,a_s$; let the multiplicity of the zero a_j be k_j $(j=1,2,\dots,s)$. If ξ is a function regular on F having at the point a_j a zero of multiplicity not less than R_j $(j=1,2,\dots,s)$, then for every $\varepsilon>0$ one can find closed sets $F_1,F_2,\dots,F_N$ and numbers $c_1,c_2,\dots,c_N$ such that

$$\bigcup_{k=1}^{n}F_k=F,\qquad F_{k'}\cap F_{k''}=\Lambda\ (k'\neq k''),\qquad \max_{z\in F}\left|\sum_{k=1}^{n}c_k[\lambda]^k(z)-\xi(z)\right|<\varepsilon.$$

Here

$$[\lambda]^k(z)=\begin{cases}0, & z\in F\setminus F_k,\\ \lambda(z), & z\in F_k.\end{cases}$$

* S. Ya. Khavinson kindly informed me that all the facts stated in § 3 had been found by him earlier from somewhat different considerations.

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Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.