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Abstract

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ON THE FLEXIBILITY OF CONVEX SURFACES WITH BOUNDARY

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In the present note we investigate the question of the flexibility of general convex surfaces homeomorphic to an open disk in the class of all convex surfaces*. For surfaces with boundary satisfying certain conditions, effectively verifiable necessary and sufficient conditions for rigidity are established:

1. Let F be a convex surface homeomorphic to an open disk, bounded by a simple closed curve Γ of bounded variation of rotation. Denote by $\overline{F}$ the boundary of the convex hull of the surface F. The complement $\overline{F} - F$ of the surface F to $\overline{F}$ is developed onto the plane, being thereby transformed into a domain Q, homeomorphic to a closed disk, generally speaking multi-sheeted. We denote the boundary of the domain Q by L. The gluing of the domain Q to the surface F, which gives the surface $\overline{F}$, is called trivial.

By virtue of A. D. Aleksandrov' s "gluing theorem" (^{1}) and A. V. Pogorelov' s theorem on the unique determination of a general closed convex surface (^{2}), the convex surface F is flexible if and only if there exists a deformation of the corresponding trivial gluing; that is, if the domain Q admits a continuous deformation preserving the conditions of gluing to the surface F.

We shall agree to denote corresponding points of the curves L and Γ under the trivial gluing by the same letters. Let the conical points of the surface $\overline{F}$ lying on the curve Γ be denoted by A_i $(i = 1, 2, ...)$, and call them vertices of type A. Let φ_{i0} be the rotation of the curve L at the point A_i from the side of the domain Q, and let φ_{i1} be the rotation of the curve Γ at the point A_i on the surface $\overline{F}$ from the side of the surface F.

Denote by Φ the set of surfaces F satisfying the conditions: 1) the boundary L of the domain Q has no points of return; 2) each point A_i of type A has on the curve L some neighborhood consisting of two rectilinear segments l_{i1} and l_{i2}, whose common endpoint is the point A.

The subset of the set Φ composed of those surfaces F to which single-sheeted domains Q correspond will be denoted by Φ'.

2. Let the convex surface $F \in \Phi'$. Denote by $\overline{Q}$ the convex hull of the domain Q. Orient the domains Q and $\overline{Q}$ by choosing, as the positive direction of

> traversal of their boundaries, that for which their interior parts lie on the left. Suppose that, when Q is traversed in the positive direction, the segment l_{i1} follows the segment l_{i2} $(i = 1, 2, ...)$.

Suppose that on the curve L there is only one point A_1 of type A, and that it is an essential vertex of the domain $\overline{Q}$. A sufficiently small neighbor- of this point on the boundary of $\overline{Q}$ consists of two rectilinear segments whose common endpoint is the point A_1. Denote these segments by l' and l'', and suppose that, in traversing $\overline{Q}$ in the positive direction, the segment l' follows l''. We shall agree to understand by the angle formed by one ray with another the angle through which the first ray must be rotated counterclockwise in order for it to coincide with the second. Denote the angle formed by the continuation of the segment l'' beyond the point A_1 with the segment l_{11} by θ_1, and the angle formed by the segment l_{12} with the continuation of the segment l' beyond the point A_1 by θ_2.

Suppose that on the curve L there is a finite or infinite number of vertices A_i of type A; moreover, if there is only one such vertex, then one and only one supporting line a to the domain Q passes through the point A_1; if there is more than one such vertex, then all of them lie on one line a supporting the domain Q. Choose on the line a a positive direction so that the domain Q lies to its right. Denote the angle formed by the negative direction of the line a with the segment l_{i1} by θ_{i1}, and the angle formed by the segment l_{i2} with the positive direction of the line a by θ_{i2}.

We single out from the set Φ' three classes of surfaces: K_1', K_2', K_3'.

To the class K_1' we assign the surfaces $F \in \Phi'$ for which the total curvature (the area of the spherical image) is equal to 4π.

To the class K_2' we assign the surfaces $F \in \Phi'$ satisfying the following conditions: 1) the curvature of the curve Γ on the surface $\overline{F}$ is concentrated at one point A_1 of type A; 2) the point A_1 is an essential vertex of the domain $\overline{Q}$; 3) at the vertex A_1 the inequalities

$$\varphi_{11} < \varphi_{10}, \qquad \tfrac{1}{2}\min(\varphi_{10} + \varphi_{11}, \pi) < \min(\theta_1, \theta_2).$$

To the class K_3' we assign the surfaces $F \in \Phi'$ satisfying the conditions: 1) the curvature of the curve Γ on the surface $\overline{F}$ is concentrated at two points A_1 and A_2 of type A; 2) the points A_1 and A_2 of the curve L lie on one line a—a supporting line of the domain Q, and on the line a there are no other points of the curve L; 3) at the vertices A_1 and A_2 the inequalities

$$\tfrac{1}{2}(\varphi_{i0} + \varphi_{i1}) < \min(\theta_{i1}, \theta_{i2}), \qquad i = 1, 2.$$

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[*] A convex surface F is called flexible in the class of all convex surfaces (below, simply flexible) if it can be included in a continuous family of isometric convex surfaces F_t not equal to it $(t \in [0, 1],\ F_0 \equiv F)$.

Theorem 1. *A convex surface $F \in \Phi'$ is non-bendable if and only if it belongs to one of the classes K_1', K_2', K_3'.*

3. Now let a convex surface $F \in \Phi$. In this case the domain Q, which completes the surface F to the surface $\overline{F}$, may turn out to be many-sheeted. Let M be some set of points in the domain Q. The set of points of the plane over which the set M is situated will be denoted by $\widetilde{M}$. The convex hull of the domain $\widetilde{Q}$ will be denoted by $\overline{Q}$. If the many-sheeted domain Q is such that all points $\widetilde{A}_i$ lie on the boundary of $\overline{Q}$ and none of them is a multiple point of the curve $\widetilde{L}$, then for it all definitions of Sec. 2 can be repeated, including the definitions of the angles θ_1, θ_2, θ_{i1}, θ_{i2}.

We single out from the set Φ three classes K_1, K_2, K_3.

To the class K_1 we assign the surfaces $F \in \Phi$ whose total curvature is equal to 4π.

To the class K_2 we assign the surfaces $F \in \Phi$ satisfying the conditions: 1) the curvature of the curve Γ on the surface $\overline{F}$ is concentrated at one point A_1 of type A; 2) the point $\widetilde{A}_1$ is not a multiple point of the curve $\widetilde{L}$; 3) the point $\widetilde{A}_1$ is an essential vertex of $\overline{Q}$; 4) at the vertex A_1 the inequalities

$$\varphi_{11} < \varphi_{10}, \qquad \tfrac{1}{2}\min(\varphi_{10} + \varphi_{11}, \pi) < \min(\theta_1, \theta_2).$$

To class K_3 we assign surfaces $F \in \Phi$ satisfying the following conditions: 1) the curvature of the curve Γ on the surface $\overline{F}$ is concentrated at a finite number $n \geqslant 1$ of points A_i $(i = 1, 2, \dots, n)$ of type A; 2) none of the points $\widetilde{A}_i$ is a multiple point of the curve $\widetilde{L}$; 3) for $n = 1$, through the vertex A_1 of the domain Q there passes one and only one supporting line a, and, whatever the points B and C of the curve L lying on the line a on different sides of the point A_1 may be, the orientations of the domains Q and $\overline{Q}$, determined by one and the same order of traversal of the points A_1, B, and C, are opposite; for $n > 1$, all vertices A_i of the domain Q lie on one of its supporting lines a; on the line a there do not exist three distinct points of the curve L, among them at least two of type A, such that the orientations of the domains Q and $\overline{Q}$, determined by one and the same order of traversal of these points, coincide; 4) at each of the vertices A_i the inequality holds

$${}^1\!/_2(\varphi_{i0} + \varphi_{i1}) < \min(\theta_{i1}, \theta_{i2}), \qquad i = 1, 2, \dots, n.$$

Theorem 2. *A convex surface $F \in \Phi$ is rigid if and only if it belongs to one of the classes K_1, K_2, K_3.*

Obviously, Theorem 1 is a particular case of Theorem 2. From Theorem 2 and the author' s work (^{3}) it follows that the necessary and sufficient conditions for rigidity of convex polyhedra homeomorphic to the open disk and bounded by a simple closed polygonal line are the same in the class of all convex polyhedra and in the class of all convex surfaces.

The proof of the necessity of the condition of Theorem 2 is carried out by means of an effective construction of a deformation of the trivial gluing. To the nine rigidity conditions that hold for polyhedra (see (3)) there are added two more: 1) on the surface $\overline{F}$ the curvature of the curve Γ, with the exception of points of type A, is not equal to zero; 2) the number of points of type A of the surface $\overline{F}$ is infinite.

Proof of sufficiency. The rigidity of surfaces $F \in K_1$ is obvious. The rigidity of surfaces $F \in K_2 + K_3$ is proved in the following way. First it is established that if a surface $F \in K_2 + K_3$, then there is no such deformation of the corresponding trivial gluing under which the turns of the curve L change at a finite number of points. This is proved separately for surfaces of the classes K_2 and K_3, and is analogous to the proof of the corresponding assertions for polyhedra in (3). Then, by means of a limiting transition, it is established that for such surfaces a deformation of the trivial gluing is altogether impossible.

In the course of the proof of Theorem 2 the following theorem is established.

Theorem 3. *A convex surface $F \in \Phi$ is a limit of convex surfaces nontrivially isometric to it if and only if it is flexible.*

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REFERENCES

[1] A. D. Alexandrov, *Intrinsic Geometry of Convex Surfaces*, Moscow-Leningrad, 1948. [2] A. V. Pogorelov, *Unique Determination of General Convex Surfaces*, Kiev, 1952. [3] L. A. Shor, *Mat. sbornik*, **45** (87), No. 4, 471 (1958).

Note: Figure translations are in progress. See original paper for figures.

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