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**Abstract**

**Full Text**

**MATHEMATICS**

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**ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH SMALL COEFFICIENTS AT THE HIGHEST DERIVATIVES IN A NEIGHBORHOOD OF AN IRREGULAR SINGULAR POINT**

*(Presented by Academician I. G. Petrovskii, 7 VII 1960)*

In the papers <sup>(1,2)</sup>, nonlinear systems of first-order differential equations with small parameters at the derivatives were investigated, and the Cauchy problem was solved. In the works <sup>(3,4)</sup>, linear equations with small parameters at the highest derivatives in a neighborhood of a regular point of the equation in a real domain were considered for nonanalytic functions, and a fundamental system of solutions was constructed. The content of <sup>(5)</sup> is the investigation of a linear equation with a small parameter at the highest derivative in a neighborhood of a regular singular point of the equation and the construction of asymptotic expansions of canonical solutions.

In the present paper we consider ordinary linear differential equations with small coefficients at the highest derivatives in a neighborhood of an irregular singular point of the first rank.

1. Consider the equation

$$\sum_{k=m+1}^n \varepsilon^{k-m} \bar{p}_k(z, \varepsilon) w^{(k)} + \sum_{k=0}^m \bar{p}_k(z, \varepsilon) w^{(k)} = 0, \quad (1)$$

where  $\bar{p}_k(z, \varepsilon)$  are analytic functions of  $z$  and  $\varepsilon$ , for which the values  $\varepsilon = 0$  and  $z = \infty$  are regular points, i.e.

$$\bar{p}_k(z, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_{k,s}(z) = \sum_{s=0}^{\infty} z^{-s} B_{k,s}. \quad (2)$$

Putting  $\varepsilon = 0$  in equation (1), we obtain the degenerate equation

$$\sum_{k=0}^m \bar{p}_k(z, 0)w^{(k)} = 0. \quad (3)$$

Let

$$\bar{p}_n(z, \varepsilon) = 1; \quad \lim_{z \rightarrow \infty} \bar{p}_m(z, 0) = B_{m,0}(0) \neq 0.$$

Under the assumptions made, the infinitely distant point is an irregular singular point of the first rank both for equation (1) and, generally speaking, for equation (3).

For the construction of formal solutions of the differential equations (1) and (3) in the form of normal series, consider the corresponding characteristic equations

$$Q(\lambda, \varepsilon) = \sum_{k=m+1}^n \lambda^k \varepsilon^{k-m} B_{k,0}(\varepsilon) + \sum_{k=0}^m \lambda^k B_{k,0}(\varepsilon) = 0; \quad (4)$$

$$Q_0(\lambda) = Q(\lambda, 0) = \sum_{k=0}^m \lambda^k B_{k,0}(0) = 0. \quad (5)$$

and the auxiliary equation

$$Q_1(\mu) = \sum_{k=0}^{n-m} \mu^k B_{k+m,0}(0) = 0. \quad (6)$$

Suppose that equations (5) and (6) have only simple roots, which we denote respectively by  $\lambda_1^0, \dots, \lambda_m^0$  and  $\mu_{m+1}^0, \dots, \mu_n^0$ ; then, for sufficiently small  $\rho = |\varepsilon|$ , equation (4) also has simple roots  $\lambda_k(\varepsilon)$ , with

$$\lambda_k(\varepsilon) = \lambda_k^0 + O(\varepsilon), \quad 1 \leq k \leq m; \quad (7)$$

$$\lambda_k(\varepsilon) = \frac{\mu_k^0}{\varepsilon} = O(1), \quad m+1 \leq k \leq n. \quad (8)$$

To each root  $\lambda_k(\varepsilon)$  of equation (4) there corresponds a formal solution of equation (1) in the form of a normal series

$$W_k(z, \varepsilon) = e^{\lambda_k(\varepsilon)z} z^{\sigma_k(\varepsilon)} \left[ 1 + \sum_{s=1}^{\infty} c_{k,s}(\varepsilon) z^{-s} \right]. \quad (9)$$

The quantities  $\sigma_k(\varepsilon)$  and  $c_{k,s}(\varepsilon)$  are determined by the known formulas for normal series. We note that, for  $1 \leq k \leq m$ ,

$$\lim_{\varepsilon \rightarrow 0} \lambda_k(\varepsilon) = \lambda_k^0, \quad \lim_{\varepsilon \rightarrow 0} \sigma_k(\varepsilon) = \sigma_k^0, \quad \lim_{\varepsilon \rightarrow 0} c_{k,s}(\varepsilon) = c_{k,s}^0. \quad (10)$$

It follows from this that the limiting normal series

$$W_k^0(z) = e^{\lambda_k^0 z} z^{\sigma_k^0} \left[ 1 + \sum_{s=1}^{\infty} c_{k,s}^0 z^{-s} \right] \quad (11)$$

is a formal solution of the degenerate equation (3). The remaining normal series (9) ( $k = m+1, \dots, n$ ) have no limit as  $\varepsilon \rightarrow 0$ . This feature in the behavior of the formal solutions of equation (1) as  $\varepsilon \rightarrow 0$  leads us naturally to pose the question of the behavior, as  $\varepsilon \rightarrow 0$ , of the true solutions of this equation.

It is known that normal series are asymptotic expansions, as  $z \rightarrow \infty$ , of true solutions of a differential equation. For every formal solution, the full neighborhood of the infinitely distant point can be divided into a number of angular domains, to each of which there belongs such a true solution of the differential equation that its asymptotic expansion as  $z \rightarrow \infty$  in the given angular domain is given by the formal solution chosen by us (6). Let us construct such domains for some formal solution of equation (1) corresponding to a root of equation (4) having a limit as  $\varepsilon \rightarrow 0$ , for example for the formal solution  $W_1(z, \varepsilon)$ .

Fix  $\arg \varepsilon = \delta_0$  and put  $\varepsilon = \rho e^{i\delta_0}$ . Introduce the notation

$$\psi_k(\rho, \delta_0) = \arg [\lambda_k(\varepsilon) - \lambda_1(\varepsilon)], \quad 2 \leq k \leq n; \quad (12)$$

$$\psi_k^0 = \arg(\lambda_k^0 - \lambda_1^0) \quad \text{for } 2 \leq k \leq m; \quad \psi_k^0 = \arg \mu_k^0 \quad \text{for } m+1 \leq k \leq n.$$

Then

$$\lim_{\rho \rightarrow 0} \psi_k(\rho, \delta_0) = \psi_k(\delta_0) = \begin{cases} \psi_k^0, & \text{if } 2 \leq k \leq m, \\ \psi_k^0 - \delta_0, & \text{if } m+1 \leq k \leq n. \end{cases}$$

Let the  $\psi_k(\delta_0)$  satisfy the conditions  $\psi_0 \leq \psi_k(\delta_0) < \psi_0 + 2\pi$ . Introduce the domains  $G_k(\rho, \delta_0)$ ,  $G_k(\delta_0)$ ,  $G_k^0$  by the following conditions:

- 1)  $z \in G_k(\rho, \delta_0)$ , if  $-\frac{2}{3}\pi - \psi_k(\rho, \delta_0) < \arg z < \frac{2}{3}\pi - \psi_k(\rho, \delta_0)$ ,
  - 2)  $z \in G_k(\delta_0)$ , if  $-\frac{2}{3}\pi - \psi_k(\delta_0) < \arg z < \frac{2}{3}\pi - \psi_k(\delta_0)$ ,
  - 3)  $z \in G_k^0$ , if  $-\frac{2}{3}\pi - \psi_k^0 < \arg z < \frac{2}{3}\pi - \psi_k^0$ .
- (13)

Define the domain  $G_0$  as the intersection of the domains  $G_k^0$ , the domain  $G(\delta_0)$ —as the intersection of the domains  $G_k(\delta_0)$ , and the domain  $G(\rho, \delta_0)$ —as the intersection of the domains  $G_k(\rho, \delta_0)$ . The size of each of the intersections is not less than  $\pi$  and not greater than  $3\pi$ . Let us note that the domains  $G_0$ ,  $G(\delta_0)$ ,  $G_k(\rho, \delta_0)$  are connected,

$$\lim_{\rho \rightarrow 0} G(\rho, \delta_0) = G(\delta_0)$$

and the domain  $G_0$  contains the domain  $G(\delta_0)$ . We shall also consider the domain  $G_\alpha(\delta_0)$ , narrower than  $G(\delta_0)$ , contained in  $G(\rho, \delta_0)$  for all sufficiently small  $\rho = |\varepsilon|$ . This domain can be obtained by bringing the boundaries of  $G(\delta_0)$  closer by some small angle  $\alpha$  on both sides.

2. For what follows it is convenient to make the substitution

$$w(z, \varepsilon) = e^{\lambda_1(\varepsilon)z} z^{\sigma_1(\varepsilon)} u(z, \varepsilon), \quad (14)$$

as a result of which equation (1) takes the form

$$\mathcal{L}[u; \varepsilon] = \sum_{k=m+1}^n \varepsilon^{k-r} p_k(z, \varepsilon) u^{(k)} + \sum_{k=0}^m p_k(z, \varepsilon) u^{(k)} = 0, \quad (15)$$

where  $p_n(z, \varepsilon)$  will be taken equal to 1,

$$p_k(z, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s a_{k,s}(z) = \sum_{s=0}^{\infty} z^{-s} b_{k,s}(\varepsilon).$$

In view of the form of the transformation (14),

$$b_{0,0}(\varepsilon) = b_{0,1}(\varepsilon) \equiv 0. \quad (16)$$

The reduced equation

$$\mathcal{L}_0[u] = \sum_{k=0}^m a_{k,0}(z) u^{(k)} = 0 \quad (17)$$

is connected with equation (3) by the transformation

$$w(z) = e^{\lambda_1^0 z} z^{\sigma_1^0} u(z). \quad (18)$$

**Lemma 1.** Let the function  $f(z)$ , as  $z \rightarrow \infty$  in the domain  $G_0$ , remain bounded:  $f(z) = O(1)$ .

Then the equation

$$\mathcal{L}_0[y] = \frac{1}{z^2} f(z) \quad (19)$$

has a unique solution  $u(z)$  which, as  $z \rightarrow \infty$  in the domain  $G_0$ , satisfies the conditions

$$u(z) = O\left(\frac{1}{z}\right); \quad u^{(k)}(z) = O\left(\frac{1}{z^2}\right), \quad k \geq 1.$$

**Lemma 2.** Let, in the nonhomogeneous equation

$$\mathcal{L}[y; \varepsilon] = \frac{1}{z^2} f(z, \varepsilon), \quad (20)$$

where the operator  $\mathcal{L}[y; \varepsilon]$  has the form (15), the function  $f(z, \varepsilon)$  remain bounded in the domain  $G_\alpha(\delta_0)$  as  $z \rightarrow \infty$ , i.e.

$$f(z, \varepsilon) = O(1). \quad (21)$$

Then equation (20) has a solution  $y(z, \varepsilon)$  which, as  $z \rightarrow \infty$  in the domain  $G_\alpha(\delta_0)$ , uniformly in  $|\varepsilon|$  for fixed  $\arg |\varepsilon| = \delta_0$ , satisfies the conditions:

$$y(z, \varepsilon) = O\left(\frac{1}{z}\right); \quad y^{(k)}(z, \varepsilon) = O\left(\frac{1}{z^2}\right), \quad k \geq 1. \quad (22)$$

This solution is unique.

The lemmas are proved by reducing the differential equations to improper integral equations of Volterra type of the second kind.

3. Let us collect in the differential operator (15)  $\mathcal{L}[y; \varepsilon]$  the terms with the same powers of  $\varepsilon$  and, isolating  $u^{(m)}$ , represent this operator in the form

$$\bar{\mathcal{L}}[u, \varepsilon] = \sum_{s=0}^{\infty} \varepsilon^s \bar{\mathcal{L}}_s[u], \quad (23)$$

where

$$\bar{\mathcal{L}}_0[u] = u^{(m)} + \sum_{k=0}^{m-1} a_{k,0}(z)u^{(k)},$$

$$\bar{\mathcal{L}}_s[u] = \sum_{k=m+1}^n a_{k,s-k+m}(z)u^{(k)} + \sum_{k=0}^{m-1} a_{k,s}(z)u^{(k)};$$

here  $a_{k,s} = 0$  for  $s < 0$ .

We shall construct a formal solution of equation (23) in the form of the series

$$U(z, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s u_s(z). \quad (24)$$

To determine the functions  $u_s(z)$  we obtain the equations

$$\bar{\mathcal{L}}[u_0] = 0, \quad (25)$$

$$\bar{\mathcal{L}}_0[u_s] = - \sum_{\mu=0}^{s-1} \bar{\mathcal{L}}_{\mu}[u_{s-1-\mu}]. \quad (26)$$

Let us choose as  $u_0(z)$  a solution of equation (25) for which, in the angular domain  $G_0$ , the asymptotic equality

$$u_0(z) \simeq U_1(z) = 1 + \sum_{s=1}^{\infty} c_{1,s}^0 z^{-s} \quad (27)$$

holds.

Successively applying Lemma 1 and the conditions (16) to equations (26), we determine all terms in the formal expansion (24). All  $u_s(z)$ ,  $s \geq 1$ , have order  $O(1/z)$  at infinity, and their derivatives decrease as  $1/z$  as  $z \rightarrow \infty$ .

On the basis of Lemma 2 the following theorem is proved:

**Theorem.** Let  $u(z, \varepsilon)$  be a solution of equation (15) having, in the domain  $G_{\alpha}(\delta_0)$ , the asymptotic expansion

$$u(z, \varepsilon) \simeq 1 + \sum_{s=1}^{\infty} c_{1,s}(\varepsilon)z^{-s}. \quad (28)$$

Let the function  $u_0(z)$  satisfy equation (25), and let the asymptotic expansion (27) hold for it in the domain  $G_0$ , while the functions  $u_s(z)$  are determined by

equations (26) and by the conditions of decrease at infinity as  $1/z$  in the domain  $G_0$ . Then the formal expansion of  $u(z, \varepsilon)$  in powers of  $\varepsilon$ , (24), will be asymptotic as  $\varepsilon \rightarrow 0$  ( $\arg \varepsilon = \delta_0$ ) in the angular domain  $G_\alpha(\delta_0)$ , so that

$$\lim_{\varepsilon \rightarrow 0} u(z, \varepsilon) = u_0(z).$$

**Remark.** The theorem formulated has been proved for a fixed argument of  $\varepsilon$ ,  $\arg \varepsilon = \delta_0$ . If  $\arg \varepsilon = \delta$  is variable,  $\delta_1 \leq \delta \leq \delta_2$ , one must take the intersection of all domains  $G(\delta)$ , which we shall denote by  $G(\delta_1, \delta_2)$ . The existence of a nonempty intersection  $G(\delta_1, \delta_2)$  is determined by the roots  $\lambda_k^0$  and by the length of the interval  $[\delta_1, \delta_2]$ . In the narrower domain  $G_\alpha(\delta_1, \delta_2)$  the formulated theorem will hold. It is impossible to construct an asymptotic expansion valid for all  $\arg \varepsilon$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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