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Abstract

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ON WEIGHTED-POLYNOMIAL APPROXIMATION OF ANALYTIC FUNCTIONS IN INFINITE DOMAINS

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The possibility of weighted-polynomial approximation of analytic functions in infinite domains was established by A. L. Shaginyan ([1,2]) and M. M. Dzhrbashyan ([3]). In application to an angular domain their result states:

For completeness of the system of polynomials in the angle Δ_α with opening $\pi\left(2-\frac{1}{\alpha}\right)\left(\frac{1}{2}<\alpha<\infty\right)$, in the presence of a normally decreasing weight $e^{-p(x)}$ ($rp'(r)\uparrow\infty$ as $r\to\infty$) in the class of functions $f(z)$ analytic in the angle and such that

$$\iint_{\Delta_\alpha} e^{-p(r)}|f(re^{i\varphi})|\,r\,dr\,d\varphi<\infty,$$

it is necessary and sufficient that

$$\int^{\infty}\frac{p(r)\,dr}{r^{1+\alpha}}$$

diverge.

In the present note we consider weighted polynomial approximation of functions analytic in infinite domains, and give an estimate for the rate of decrease of the best approximations depending on the properties of the approximated function and of the weight.

Theorem. Let $f(z)$ be a function analytic in the half-plane $\operatorname{Im} z\geqslant 0$; $|f(z)|\leqslant M(r)$; $|z|\leqslant r$; $\operatorname{Im} z\geqslant 0$; Δ_l is the half-plane $\operatorname{Im} z\geqslant l$. If we denote

$$E_n(f,h,l)=\inf_{\{P_n(z)\}}\sup_{\operatorname{Im} z\geqslant l} h(z)\,|f(z)-P_n(z)|,$$

then

$$E_n(f, e^{-|z|^\lambda}, l) \leqslant CM(n^{1/\lambda}) \exp\{-ln^{(\lambda-1)/\lambda}\} \quad \text{for } \lambda > 1,$$

where C does not depend on n.

For the proof, denote by C_R the semicircle $|z| = R$, $\operatorname{Im} z \geqslant 0$; by L_R the segment $-R \leqslant z \leqslant R$, and, following S. N. Mergelyan ([4]), form the function

$$G_R(z) = \frac{1}{2\pi i} \int_{L_R} f(\zeta) V_\zeta(z)\, d\zeta + \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)\, d\zeta}{\zeta - z},$$

where

$$V_\zeta(z) = \frac{e^{im_\zeta z} - e^{im_\zeta \zeta}}{(z-\zeta) e^{im_\zeta z}}; \qquad m_\zeta = \left[\frac{1}{l} \ln \frac{M(|\zeta|)(1+|\zeta|^2)}{2\delta l}\right].$$

It is easy to show that the functions $G_R(z)$, as $R \to \infty$, converge uniformly in every finite part of the plane to a certain entire function. Indeed, let $|z| < R < R'$; then, by virtue of the analyticity of $f(z)$ in $\operatorname{Im} z \geqslant 0$

and the closeness of $V_\zeta(z)$ to $(\zeta - z)^{-1}$ on the axis $-\infty < x < \infty$, we have

$$|G_{R'}(z) - G_R(z)| \leq \frac{1}{2\pi} \int_{L_{R'} - L_R} |f(\zeta)| \left| V_\zeta(z) - \frac{1}{\zeta - z} \right| d\zeta \leq$$
$$\leq \frac{1}{2\pi} \int_{L_{R'} - L_R} \frac{1}{l} M(R) e^{-mRl}\, dR \leq \frac{1}{2\pi} \int_{L_{R'} - L_R} \frac{dR}{1+R^2} \to 0 \quad \text{as } R \to \infty.$$

Denote by $G(z)$ the entire function $\lim_{R\to\infty} G_R(z)$. We have in Δ_l

$$|G(z) - f(z)| = \lim_{x\to\infty} |G_x(z) - f(z)| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(\zeta)| \left| V_\zeta(z) - \frac{1}{\zeta - z} \right| d\zeta \leq \delta.$$

Let us estimate the growth of the function $G(z)$; for this purpose, choosing $R > 2|z|$, we write

$$G_R(z) = \frac{1}{2\pi i} \int_{C_{2|z|}} \frac{f(\zeta)\, d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{L_R - L_{2|z|}} f(\zeta) \left(V_\zeta(z) - \frac{1}{\zeta - z} \right) d\zeta +$$
$$+ \frac{1}{2\pi i} \int_{L_{2|z|}} f(\zeta) V_\zeta(z)\, d\zeta = I_1 + I_2 + I_3.$$

We have $|I_1| \leq M(2|z|)$; $|I_2| \leq \delta$; taking into account that $V_\zeta(z) \leq C_1 e^{4m_\zeta |z|}$, we obtain for the third term

$$|I_3| \le \frac{2|z|}{\pi} M(2|z|) \exp\left\{ \frac{|z|}{l} \ln \frac{M(2|z|)(1+4|z|^2)}{2\delta l} \right\}.$$

Thus,

$$G(r) \le c \exp \frac{r}{l} \ln \frac{M(2r)(1+4r^2)}{\delta}.$$

Now, if $P_n(z) = \sum_0^n a_k z^k$ is a segment of the Taylor series about the origin of the entire function $G(z)$, then for $|z| \le R/e$ in Δ_l, when

$$n = \left[C \frac{R}{l} \ln \frac{M(2R)}{\delta} \right]$$

(R so far arbitrary), $|P_n(z) - f(z)| \le 2\delta$.

In the half-plane Δ_l outside the circle $|z| \le R/e$, the closeness is ensured by the weight function $e^{-|z|^\lambda}$, $\lambda > 1$.

With a suitable choice of R from the expression for the degree of the approximating polynomial, we obtain

$$E_n\left(f, e^{-|z|^\lambda}\right) \le \delta = CM(n^{1/\lambda}) \exp\left(- \ln \frac{\lambda - 1}{\lambda} \right).$$

The case considered here, for simplicity of exposition, of a half-plane and of the weight $e^{-|z|^\lambda}$, $\lambda > 1$, by the methods which we used in ([5]), is easily generalized to the case of a broader class of infinite domains and more general weights.

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CITED LITERATURE

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4. S. N. Mergelyan, UMN, 7, 2, 48 (1952).

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Note: Figure translations are in progress. See original paper for figures.