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Abstract

Full Text

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Semidirect Products of Cyclic p -Groups

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The work is devoted to p -groups that are decomposable into a semidirect product of cyclic subgroups. Groups of this kind can be characterized as p -groups possessing an invariant series with cyclic factors, all members of which are complemented in the group. In the commutative case this condition is satisfied by direct products of cyclic p -groups and only by them. In the general case the p -groups under consideration turn out to be two-step solvable (see Theorem 3).

It is known that in the commutative case of the p -groups considered here, any two of their direct decompositions with cyclic factors are isomorphic ⁽¹⁾; in the general case it turns out that this property is preserved for $p \neq 2$ for any two semidirect decompositions of this kind (see Theorem 9).

It is known that every subgroup of a commutative group of the class of p -groups under consideration belongs to this class ⁽¹⁾. In the case of noncommutative p -groups decomposable into a semidirect product of cyclic subgroups, this property is lost (see Example 2); however, for abelian subgroups of such groups (for $p \neq 2$) it is nevertheless preserved (see Theorem 10).

In order to describe the class of p -groups decomposable into a semidirect product of cyclic subgroups, it proved necessary to define the external semidirect product of cyclic p -groups, i.e., the semidirect product of arbitrary cyclic p -groups under prescribed admissible relations between their elements (see Theorem 7). It turned out that the class of p -groups representable in the form of a semidirect product of cyclic subgroups coincides with the class of all possible external semidirect products of cyclic p -groups (see Theorem 8).

- Definition 1.** We shall call a group \mathfrak{G} the **semidirect product of its subgroups** $\mathfrak{A}_1, \dots, \mathfrak{A}_\alpha, \dots$ ($\alpha \in \mathfrak{M}, \mathfrak{M}$ —some set of ordinal numbers ν satisfying one of two conditions: $1 \leq \nu < \gamma$ or $1 \leq \nu \leq \gamma$) and shall write

$$\mathfrak{G} = [\mathfrak{A}_1 \dots \mathfrak{A}_\alpha \dots] \quad (\alpha \in \mathfrak{M}), \quad (1)$$

if the following requirements are fulfilled: 1) the group \mathfrak{G} is generated by the subgroups $\mathfrak{A}_1, \dots, \mathfrak{A}_\alpha, \dots$; 2) if α is an arbitrary ordinal number from \mathfrak{M} distinct from one, then the subgroup \mathfrak{G}_α , generated by the subgroups $\mathfrak{A}_\beta, \beta < \alpha$ ($\beta \in \mathfrak{M}$), is invariant in the group \mathfrak{G} and has trivial intersection with the subgroup $\mathfrak{G}^{(\alpha)}$, generated by the subgroups $\mathfrak{A}_\nu, \nu \geq \alpha$ ($\nu \in \mathfrak{M}$).

We shall call the group \mathfrak{G} **decomposable into a semidirect product of its subgroups** \mathfrak{A}_α ($\alpha \in \mathfrak{M}$), and these subgroups \mathfrak{A}_α the **factors** of the given decomposition.

In the paper ⁽²⁾ some properties of groups decomposable into a semidirect product of their subgroups, following from the above definition, are noted, and, in particular, the following: a) all factors of the decomposition (1) are pairwise permutable with one another; b) the intersection of any subgroup

\mathfrak{A}_α ($\alpha \in \mathfrak{M}$) with the subgroup generated by all \mathfrak{A}_β , $\beta \neq \alpha$ ($\beta \in \mathfrak{M}$), is equal to the identity;

c) for any two factors \mathfrak{A}_α and \mathfrak{A}_β , $\alpha < \beta$, of the decomposition (1), $\{\mathfrak{A}_\alpha, \mathfrak{A}_\beta\} = [\mathfrak{A}_\alpha \mathfrak{A}_\beta]$.

Theorem 1. If $\mathfrak{G} = [\{A\}\{B\}]$ is some decomposition of a p -group \mathfrak{G} into a semidirect product of cyclic subgroups with relations

$$A^{p^m} = B^{p^n} = E, \quad BAB^{-1} = A^s, \quad A^s \neq A,$$

then its commutator subgroup coincides with the subgroup $\{A^{s-1}\}$; this subgroup is distinct from $\{A\}$, and therefore $s = Np^k + 1$, where $(N, p) = 1$, $1 \leq k < m$.

Lemma 1. If

$$\mathfrak{G} = [\{A_1\} \dots \{A_\alpha\} \dots] \quad (\alpha \in \mathfrak{M}) \quad (2)$$

is a decomposition of the p -group \mathfrak{G} into a semidirect product of cyclic subgroups with relations

$$A_\alpha^{p^{m_\alpha}} = E, \quad A_\beta A_\alpha A_\beta^{-1} = A_\alpha^{s_{\alpha\beta}}, \quad \alpha < \beta \quad (\beta \in \mathfrak{M}), \quad (3)$$

where $s_{\alpha\beta} = N_{\alpha\beta} p^{k_{\alpha\beta}} + 1$ ($(N_{\alpha\beta}, p) = 1$, $k_{\alpha\beta} \geq 1$ or $N_{\alpha\beta} = 0$), then the congruences

$$s_{\alpha\beta}^{p^{m_\beta}} \equiv 1 \pmod{p^{m_\alpha}}, \quad s_{\alpha\beta}^{s_{\beta\gamma} - 1} \equiv 1 \pmod{p^{m_\alpha}}, \quad \beta < \gamma \quad (\gamma \in \mathfrak{M})$$

are satisfied.

If the subgroup $[\{A_\alpha\}\{A_\beta\}]$ of the decomposition (2) is abelian, then we shall set the number $k_{\alpha\beta}$ in the relations (3) equal to m_α .

Theorem 2. The lower layer \mathfrak{G}_1 of the p -group \mathfrak{G} (i.e. the set of its elements of order not exceeding p) with decomposition (2) and relations (3), for $p \neq 2$, coincides with the subgroup $\{A_1^{p^{m_1-1}}\} \times \dots \times \{A_\alpha^{p^{m_\alpha-1}}\} \times \dots$.

Remark. For 2-groups of this kind, Theorem 2 does not hold.

Example 1. In the group $\mathfrak{G} = [\{A\}\{B\}]$ with relations $A^8 = B^2 = E$, $BAB^{-1} = A^3$ (the existence of such a group follows from Theorem 7 given below), the lower layer consists of its following 6 elements: $E, A^4, B, A^2B, A^4B, A^6B$. If Theorem 2 were valid in the case of the group under consideration, then its lower layer would have to contain only the elements E, A^4, B, A^4B .

2. Theorem 3. The commutator subgroup \mathfrak{G}' of the p -group \mathfrak{G} with decomposition (2) and relations (3) coincides with the subgroup

$$\{A_1^{p^{\bar{k}_1}}\} \times \dots \times \{A_\alpha^{p^{\bar{k}_\alpha}}\} \times \dots,$$

where $\bar{k}_\alpha = \min(k_{\alpha\alpha+1}, \dots, k_{\alpha\beta}, \dots)$, $\alpha < \beta$, and each subgroup $\{A_\alpha^{p^{\bar{k}_\alpha}}\}$ is invariant in the group \mathfrak{G} .

Theorem 4. If

$$\mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \mathfrak{G}_2 \supseteq \dots \supseteq \mathfrak{G}_n \supseteq \dots$$

is the lower central series of the p -group \mathfrak{G} with decomposition (2) and relations (3), then

$$\mathfrak{G}_n = \{A_1^{p^{\bar{k}_1 n}}\} \times \dots \times \{A_\alpha^{p^{\bar{k}_\alpha n}}\} \times \dots, \quad n = 0, 1, 2, \dots,$$

where $\bar{k}_\alpha = \min(k_{\alpha\alpha+1}, \dots, k_{\alpha\beta}, \dots)$, $\beta > \alpha$ ($\beta \in \mathfrak{M}$).

Theorem 5. A p -group \mathfrak{G} with decomposition (2) and relations (3) is nilpotent if and only if for the numbers $k_1, k_2, \dots, \bar{k}_\alpha, \dots$, where $k_\alpha = \min(k_{\alpha\alpha+1}, \dots, k_{\alpha\beta}, \dots)$, $\beta > \alpha$ ($\beta \in \mathfrak{M}$), there exists a natural number M such that for every α the inequality $m_\alpha \leq \bar{k}_\alpha M$ holds.

Corollary 1. If the orders of all factors of the decomposition (2) of the p -group \mathfrak{G} are bounded in the aggregate, then the group \mathfrak{G} is nilpotent.

3. Theorem 6. A mapping φ of the p -group \mathfrak{G} with decomposition (2) and relations (3) onto itself, defined by the following two conditions, is an automorphism of this group:

- 1) for all elements A_α^x of $\{A_\alpha\}$ ($\alpha \in \mathfrak{M}$)

$$A_\alpha^x \varphi = A_\alpha^{x r_{\alpha\varphi}},$$

where $r_{\alpha\varphi}$ is some natural number of the form $r_{\alpha\varphi} = M_{\alpha\varphi} p^{q_{\alpha\varphi}} + 1$, with $(M_{\alpha\varphi}, p) = 1$, $q_{\alpha\varphi} \geq 1$, or $M_{\alpha\varphi} = 0$, satisfying the relation

$$s_{\gamma\alpha}^{r_{\alpha\varphi}-1} \equiv 1 \pmod{p^{m_\gamma}}, \quad \gamma < \alpha \ (\gamma \in \mathfrak{M});$$

- 2) for every element $g = A_{\beta_1}^{i_1} A_{\beta_2}^{i_2} \dots A_{\beta_t}^{i_t} \in \mathfrak{G}$, $\beta_1 < \beta_2 < \dots < \beta_t$
 $(\beta_1, \dots, \beta_t \subset \mathfrak{M})$

$$g\varphi = (A_{\beta_1}^{i_1})\varphi(A_{\beta_2}^{i_2})\varphi \dots (A_{\beta_t}^{i_t})\varphi.$$

Theorem 7. Let $\{A_\alpha\}$ ($\alpha \in \mathfrak{M}$) be arbitrary cyclic p -groups of orders p^{m_α} , and let the numbers $s_{\alpha\beta}$, $\beta > \alpha$ ($\beta \in \mathfrak{M}$), satisfy the conditions:

- 1) $s_{\alpha\beta} = N_{\alpha\beta} p^{k_{\alpha\beta}} + 1$, where $(N_{\alpha\beta}, p) = 1$, $k_{\alpha\beta} \geq 1$, or $N_{\alpha\beta} = 0$;
- 2) $s_{\alpha\beta}^{s_{\beta\gamma}^{-1}} \equiv 1 \pmod{p^{m_\alpha}}$, $\gamma > \beta$ ($\gamma \in \mathfrak{M}$);
- 3) $s_{\alpha\beta}^{p^{m_\beta}} \equiv 1 \pmod{p^{m_\alpha}}$.

Then the group \mathfrak{G} (we shall call it an outer semidirect product of the cyclic p -groups $\{A_\alpha\}$), generated by the elements of all groups $\{A_\alpha\}$ subject to the relations

$$A_\beta A_\alpha A_\beta^{-1} = A_\alpha^{s_{\alpha\beta}}, \quad \alpha < \beta \tag{4}$$

(the relations between the elements of each group $\{A_\alpha\}$ are preserved also in the group \mathfrak{G}), decomposes into the semidirect product

$$\mathfrak{G} = [\{A_1\} \dots \{A_\alpha\} \dots],$$

satisfying the relations (4).

Remark. In view of a certain arbitrariness in the choice of the numbers $s_{\alpha\beta}$, for one and the same cyclic p -groups $\{A_\alpha\}$ ($\alpha \in \mathfrak{M}$) one can construct a series of outer semidirect products that are not isomorphic to one another.

From Theorem 7 and Lemma 1 one can derive the following proposition.

Theorem 8. The class of p -groups decomposable into a semidirect product of cyclic subgroups coincides with the class of all possible outer semidirect products of cyclic p -groups.

4. Definition 2. Two decompositions of a p -group into a semidirect product of cyclic subgroups will be called **isomorphic** if between their factors one can establish such a one-to-one correspondence under which the corresponding factors are isomorphic groups.

Theorem 9. If a p -group \mathfrak{G} is decomposable into a semidirect product of cyclic subgroups, then for $p \neq 2$ any two such decompositions of it are isomorphic to one another.

Theorem 10. Every abelian subgroup of a p -group \mathfrak{G} , decomposable into a semidirect product of cyclic subgroups, for $p \neq 2$, is decomposable into a direct product of cyclic subgroups.

Remark 1. We shall call a noncommutative 2-group \mathfrak{G} with a decomposition (2) a **special group** if in this decomposition there are such factors $\{A_\alpha\}$ and $\{A_\beta\}$, $\beta > \alpha$ ($\beta \in \mathfrak{M}$), that the elements A_α and A_β of the noncommutative group $[\{A_\alpha\}\{A_\beta\}]$ are connected by the relation $A_\beta A_\alpha A_\beta^{-1} = A_\alpha^{N \cdot 2 + 1}$, where $(N, 2) = 1$. Theorems 2, 9, and 10 are also valid for 2-groups with decomposition (2) when they are not special. The case of special groups requires additional investigations.

Remark 2. Not every subgroup of a p -group ($p \neq 2$) with decomposition (2) is decomposable into a semidirect product of cyclic subgroups.

Example 2. In the p -group $\mathfrak{G} = [\{A\}\{B\}]$ with relations $A^{p^5} = B^{p^4} = E$ ($p \neq 2$) and $BAB^{-1} = A^{p^2+1}$ (the existence of such a group follows from Theorem 7), the subgroup $\mathfrak{A} = \{A^{p^2}\}\{AB\}$ does not decompose into a semidirect product of cyclic subgroups.

The topic of the present work was proposed to the author by S. N. Chernikov and was developed under his supervision.

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Note: Figure translations are in progress. See original paper for figures.

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