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Abstract

Full Text

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On the Concept of a Normal Scale of Spaces

(Presented by Academician N. N. Bogolyubov, 28 I 1960)

In the note ⁽¹⁾ we introduced for consideration the concept of an analytic scale of spaces as a family of Banach spaces obtained by a special construction. The norms in these spaces depend analytically on the parameter. In the present article we consider a more general definition of a scale of spaces, in which, instead of analyticity, only logarithmic convexity is required of the dependence of the norms on the parameter.

1°. **Definition.** A family of Banach spaces E_α ($\alpha_0 \leq \alpha \leq \beta_0$) with norms $\|x\|_\alpha$ is called a **normal scale of spaces** if the following conditions are satisfied:

1. For $\beta > \alpha$, the set E_β is contained in the space E_α , is dense in it, and

$$\|x\|_\alpha \leq \|x\|_\beta \quad \text{for } x \in E_\beta. \quad (1)$$

2. If $\alpha_0 \leq \alpha \leq \beta \leq \gamma \leq \beta_0$ and $x \in E_\gamma$, then

$$\|x\|_\beta \leq \|x\|_\alpha^{\frac{\gamma-\beta}{\gamma-\alpha}} \|x\|_\gamma^{\frac{\beta-\alpha}{\gamma-\alpha}}. \quad (2)$$

As was indicated in ⁽¹⁾, property (2) is satisfied for spaces of an analytic scale. In order that (1) also be satisfied, it is necessary to impose an inessential additional restriction on the operators $T(z)$. The classical scales of spaces considered in analysis, after a certain renormalization, become normal. We give examples.

In an n -dimensional domain G , consider the spaces L_p of functions summable with the p -th power ($p \geq 1$). Denote by L^α the space L_p for

$$p = \frac{2}{1-\alpha}$$

with norm

$$\|f\|_\alpha = (\text{mes } G)^{\frac{\alpha-1}{2}} \|f\|_{L_p} \quad \left(p = \frac{2}{1-\alpha}, f \in L_p \right). \quad (3)$$

The spaces L^α form a normal scale for $-1 \leq \alpha \leq 1$. A normal scale for $0 \leq \alpha \leq 1$ is formed by the spaces C_α of functions continuous in \bar{G} and satisfying in G a Hölder condition, with norm

$$\|f\|_{C_\alpha} = \max \left\{ \max_{x \in \bar{G}} |f(x)|, D^\alpha \sup_{x, y \in G} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\}, \quad (4)$$

where D is the diameter of the domain G .

Finally, a normal scale for $-\infty < \alpha < \infty$ is formed by the spaces $L_{p,\alpha}$ (see (1)) with norm

$$\|f\|_{L_{p,\alpha}} = D^\alpha \left\{ \int \frac{|f|^p}{r^{p\alpha}} dx \right\}^{1/p}, \quad (5)$$

where r is the distance from the point x to some fixed point of \bar{G} or to some manifold lying in \bar{G} .

Let us indicate the simplest properties of normal scales. If, instead of the index α , one introduces an index α' linearly related to α : $\alpha = k\alpha' + \theta$ ($k > 0$), and denotes $\tilde{E}_{\alpha'} = E_{k\alpha' + \theta} = E_\alpha$, then the spaces $\tilde{E}_{\alpha'}$ form an α_0 -normal scale on the interval

$$\left[\frac{\alpha_0 - \theta}{k}, \frac{\beta_0 - \theta}{k} \right].$$

Thus, if α_0 and β_0 are finite, then without loss of generality they may be assumed equal to 0 and 1.

If in the spaces E_α of a normal scale one makes an equivalent renorming, then, generally speaking, inequalities (1) and (2) are violated. However, there exists a class of renormings that does not violate (1) and (2). Let $q(\alpha)$ be a nondecreasing logarithmically convex function on the interval $[\alpha_0, \beta_0]$; then the spaces E_α form a normal scale with respect to the norms $\|x\|_{\alpha,1} = q(\alpha)\|x\|_\alpha$.

Let E_α ($\alpha_0 \leq \alpha \leq \beta_0$) be a family of Banach spaces forming a normal scale on the intervals $[\alpha_0, \delta]$ and $[\delta, \beta_0]$, where $\alpha_0 < \delta < \beta_0$. In order that these spaces form a normal scale on the whole interval $[\alpha_0, \beta_0]$, it is necessary and sufficient that inequality (2) hold for $\beta = \delta$ and arbitrary $\alpha < \delta$ and $\gamma > \delta$ ($\alpha, \gamma \in [\alpha_0, \beta_0]$).

If the interval $[\alpha_0, \beta_0]$ is covered by intervals $[\alpha_0, \beta_1), (\beta_1, \beta_2), \dots, (\beta_{k-1}, \beta_0)$, and the family of Banach spaces E_α ($\alpha_0 \leq \alpha \leq \beta_0$) forms a normal scale on each of the intervals $[\alpha_0, \beta_1), (\beta_1, \beta_2), \dots, (\beta_{k-1}, \beta_0)$, then the family E_α will be a normal scale of spaces on the entire interval $[\alpha_0, \beta_0]$.

If E_α is a normal scale on $[\alpha_0, \beta_0]$ and F_α is a normal scale on $[\delta, \beta_0]$ ($\alpha_0 < \delta < \beta_0$) such that $F_\delta = E_\delta$, $F_\alpha \subset E_\alpha$, and $\|x\|_{E_\alpha} \leq \|x\|_{F_\alpha}$ ($x \in F_\alpha$) for $\delta < \alpha \leq \beta_0$, then the spaces \tilde{E}_α , coinciding with E_α for $\alpha \in [\alpha_0, \delta]$ and with F_α for $\alpha \in [\delta, \beta_0]$, form a normal scale on $[\alpha_0, \beta_0]$.

A widespread construction of a normal scale is the following: one considers a certain linear set M , and on it specifies a family of norms $\|x\|_\alpha$ ($x \in M$, $\alpha_0 \leq \alpha \leq \beta_0$) satisfying conditions (1) and (2).

If E_α denotes the completion of M in the norm $\|x\|_\alpha$ for $\alpha < \beta_0$, and E_{β_0} denotes the completion of M in the norm

$$\|x\|_{\beta_0} = \lim_{\alpha \rightarrow \beta_0} \|x\|_\alpha,$$

then the spaces E_α will form a normal scale on the interval $[\alpha_0, \beta_0]$. If, however, \widetilde{E}_{β_0} denotes the completion of M in the norm $\|x\|_{\beta_0}$, which, generally speaking, is larger than $\|x\|_{\beta_0}$, then \widetilde{E}_{β_0} may fail to be contained in the spaces E_α for $\alpha < \beta_0$. However, there exists a homomorphic mapping of \widetilde{E}_{β_0} into E_α , and if one replaces \widetilde{E}_{β_0} by the quotient space E'_{β_0} by the kernel of this homomorphism, then the resulting family of spaces (E_α for $\alpha < \beta_0$ and E'_{β_0}) will form a normal scale on $[\alpha_0, \beta_0]$. If one constructs the scale L^α by completing a certain set of bounded functions, defining $\|f\|_\alpha$ for $\alpha < 1$ by formula (3) and putting $\|f\|_1 = \sup |f(x)|$, then the passage from the norm $\|f\|_{\beta_0}$ to the norm $\|f\|'_{\beta_0}$ corresponds to the passage from the norm $\|f\|_1$ to the norm $\|f\|'_1 = \text{vraisup} |f(x)|$. This passage to the quotient space corresponds to identifying functions that differ on a set of measure zero.

2°. Consider two Banach spaces F_0 and F_1 such that

$$F_1 \subset F_0, \quad F_1 \text{ is dense in } F_0, \quad \|x\|_{F_0} \leq \|x\|_{F_1} \quad (x \in F_1). \quad (6)$$

Suppose that on the interval $[0, 1]$ a normal scale of spaces E_α is given such that

$$F_0 \subset E_0, \quad F_1 \subset E_1, \quad \|x\|_{E_0} \leq \|x\|_{F_0} \quad (x \in F_0), \quad \|x\|_{E_1} \leq \|x\|_{F_1} \quad (x \in F_1); \quad (7)$$

$$\lim_{\beta \rightarrow 1} \|x\|_{E_\beta} = \|x\|_{E_1} \quad (x \in E_1). \quad (8)$$

We shall say that the scale E_α is **based on the spaces** F_0 and F_1 . One scale based on F_0 and F_1 exists—this is the trivial scale $E_\alpha \equiv F_0$.

Theorem 1. *Among the normal scales based on the spaces F_0 and F_1 , there exists a scale G_α ($0 \leq \alpha \leq 1$) having the following properties: a) $G_0 = F_0$; b) F_1 is dense in G_1 ; c) for any normal scale E^α based on F_0 and F_1 , the inequality*

$$\|x\|_{E_\alpha} \leq \|x\|_{G_\alpha} \quad (0 \leq \alpha \leq 1, x \in F_1)$$

holds.

The scale whose existence is asserted in the theorem will be called the **maximal scale constructed from the spaces F_0 and F_1** . If there exists some scale based on F_0 and F_1 such that $E_1 = F_1$, then for the maximal scale we shall also have $G_1 = F_1$. In this case the spaces F_0 and F_1 will be called **related**.

Remark. If G_α ($0 \leq \alpha \leq 1$) is the maximal scale constructed from the spaces F_0 and F_1 , then on any interval $[\alpha_0, \beta_0] \subset [0, 1]$ it will be the maximal scale constructed from the spaces G_{α_0} and G_{β_0} .

Theorem 2 (interpolation). Let F_0 and F_1 be two spaces satisfying conditions (6); let G_α ($0 \leq \alpha \leq 1$) be the maximal scale constructed from them; and let E_α be an arbitrary normal scale on $[0, 1]$ satisfying condition (8).

If a linear operator A is defined on F_1 and satisfies the conditions

$$\|Ax\|_{E_0} \leq C_0 \|x\|_{F_0}, \quad \|Ax\|_{E_1} \leq C_1 \|x\|_{F_1} \quad (x \in F_1),$$

then for $0 \leq \alpha \leq 1$ the inequality

$$\|Ax\|_{E_\alpha} \leq C_0^{1-\alpha} C_1^\alpha \|x\|_{G_\alpha} \quad (x \in F_1) \quad (9)$$

holds.

Theorem 2 is naturally reformulated for the case when the scale G_α is defined on the interval $[\alpha_0, \beta_0]$, and the scale E_α on the interval $[\alpha_1, \beta_1]$.

We note that in the proof of the theorem one uses not the fact that the operator A is linear, but only the fact that the quantity $\|Ax\|_\alpha$ has the properties of a norm.

3°. Consider, as an example, the spaces L_1 and L_∞ of functions defined on the interval $[0, 1]$. These spaces satisfy conditions (6) and are related, since the scale of spaces L^α is based on them (see (3)), and $L^1 = L_\infty$. It would have been natural to suppose that the scale L^α is maximal. However, this supposition is false. As E. M. Semenov showed jointly with the author, the maximal scale constructed from the spaces L_1 and L_∞ is the scale of spaces S_α , consisting of summable functions $f(t)$ on $[0, 1]$ for which

$$\|f\|_{S_\alpha} = (1 - \alpha) \int_0^1 t^{-\alpha} f^*(t) dt < \infty, \quad (10)$$

where by $f^*(t)$ is denoted the rearrangement of the function $|f(t)|$ in decreasing order (see (2), p. 332).

Similarly, for the space V_α of absolutely continuous functions on $[0, 1]$ with norm $\|f\|_{V_\alpha} = \max\{\max|f(t)|, \text{Var } f(t)\}$ and the space C_1 of functions satisfying the

Lipschitz condition, with norm (4), the maximal scale based on them is the scale of spaces S_α^1 , consisting of absolutely continuous functions for which

$$\|f\|_{S_\alpha^1} = \max \left\{ \max |f(t)|, (1 - \alpha) \int_0^1 t^{-\alpha} f^*(t) dt \right\} < \infty. \quad (11)$$

From Theorem 2 one can obtain a number of theorems on operators defined in the spaces S_α and S_α^1 .

Let us formulate some of them:

Theorem 3. *If the operator A is a bounded operator acting from the space L_1 into the space R_α and from the space L_∞ into the space R_β , where R_α and R_β are two spaces either from the scale L^α (see (3)), or from the scale C_α (see (4)), or $L_{p,\beta}$ (see (5)), or S_α (see (10)), or S_α^1 (see (11)), with the corresponding indices, then the operator A is a bounded operator acting from the space S_μ ($0 \leq \mu \leq 1$) into the space $R_{\alpha(\mu)}$, where $\alpha(\mu) = \alpha(1 - \mu) + \beta\mu$.*

Using the remark to Theorem 2, one can prove a more general theorem:

Theorem 4. *If the operator A is a bounded operator acting from the space $S_{\alpha'}$ into the space R_α and from $S_{\beta'}$ into R_β , where R_α and R_β are the same as in Theorem 3, then the operator A is a bounded operator acting from the space $S_{\alpha'(\mu)}$ into the space $R_{\alpha(\mu)}$, where $\alpha'(\mu) = \alpha'(1 - \mu) + \beta'\mu$, $\alpha(\mu) = \alpha(1 - \mu) + \beta\mu$ ($0 \leq \mu \leq 1$).*

Using inequality (9), in Theorems 3 and 4 one can estimate the norms of the operator A in the corresponding spaces.

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References

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- ² G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, IL, 1948.

Note: Figure translations are in progress. See original paper for figures.

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