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V. P. PALAMODOV

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Abstract

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MATHEMATICS

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ON CONDITIONS FOR CORRECT SOLVABILITY IN THE LARGE OF A CERTAIN CLASS OF EQUATIONS WITH CONSTANT COEFFICIENTS

(Presented by Academician P. S. Aleksandrov on 21 I 1960)

In the present note equations of the form

$$p\left(i\frac{\partial}{\partial x}\right)u = w, \tag{1}$$

are considered, where $p(s) = p(s_1, \dots, s_n)$ is a polynomial in n complex variables $s_j = \sigma_j + i\tau_j,\ 1 \le j \le n$, which does not vanish on the real manifold, i.e., for $\tau_1 = \dots = \tau_n = 0$ (or a matrix whose entries are polynomials with det $\ne 0$).

The problem is posed as follows: to find, in terms of smoothness and growth at infinity, the classes of uniqueness and the classes of right-hand sides w for which there exists a solution of the equation satisfying one or another restriction imposed on its growth. For $n > 1$ we introduce the number γ—the genus of the equation (system), equal to the least upper bound of those γ such that, for some c, in the domain

$$T(c,\gamma) = \{\sigma + i\tau:\ |\tau| \le c|\sigma|^\gamma\};$$

$$|\sigma| = |\sigma_1| + \dots + |\sigma_n|; \quad |\tau| = |\tau_1| + \dots + |\tau_n|,$$

for sufficiently large $|\sigma|$ the polynomial $p(s)$ continues not to vanish*. For $n = 1$ one may henceforth take $\gamma = 1$. In general it is not known whether such a domain $T(c,\gamma)$ exists with γ equal to the genus of the equation; therefore in what follows by γ we shall mean any number less than the genus.

In the case $\gamma > 0$, the results of the present work are an extension of certain results obtained in ([3]).

The reasoning is based on the following estimate:

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$$\left|D^q \frac{1}{p(\sigma)}\right| \leq A_\rho B_\rho^{|q|}\, |q|^{|q|}\, |\sigma + i|^{-\gamma|q|+m}; \qquad D^q = \frac{\partial^q}{\partial s_1^{q_1} \cdots \partial s_n^{q_n}};$$

$$|q| = |q_1| + \cdots + |q_n|$$

(m is the order of $p(s)$), in view of the fact that finding the classes of uniqueness and the classes of existence reduces to studying the operator of multiplication by the function $\dfrac{1}{p(\sigma)}$ in various spaces.

Using this estimate and the technique of spaces of type S (the definition and properties of these spaces are set forth in ([1]), Chap. 4), we find classes

* Formally speaking, for the value γ the quantity $-\infty$ is also admissible; the subsequent results remain valid also in this case. But, using the methods of Seidenberg–Tarski ([2]), it can be shown that always $\gamma > -\infty$.

uniqueness. In the case $\gamma \geqslant 0$, the solution of equation (1) will be unique only in the space $(S^\beta_{1,A})'$, for $\beta > 0$ and some A; in particular, it is unique in the class of functions satisfying the inequality

$$|u(x)| \leqslant C \exp\left[\sum a_j |x_j|\right], \qquad (x = (x_1, \dots, x_n)),$$

where it turns out that one may put

$$a_j = \inf_{p(s)=0} |\tau_j| - \varepsilon,$$

the lower bound being taken over all roots of the polynomial $p(s)$; but if one puts

$$a_j = \inf_{p(s)=0} |\tau| + \varepsilon,$$

then uniqueness will be violated.

In the case $\gamma < 0$, as a uniqueness class one may take any space $(S^\beta_\alpha)'$, if $\alpha + \gamma\beta \geqslant 1$ and $\alpha > 1$; in particular, uniqueness will hold in the class of functions satisfying the inequality

$$|u(x)| \leqslant C \exp[a|x|^{1-\varepsilon}].$$

for any positive a and ε. At the same time, uniqueness will already be violated in the class of functions satisfying this inequality with $\varepsilon = 0$ and arbitrarily small a.

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To find existence classes we shall need the following spaces:

$$E_{\pm\alpha,A} = \left\{\chi(x) : \bar{\chi}(x) \exp\left[\pm\frac{1}{A}|x|^{1/\alpha}\right] \in L_2\right\} \qquad (A > 0);$$

$$H_{(\pm k)} = \{\chi(x) : \chi(x)|x+i|^{\pm k} \in L_2\} \qquad (k > 0).$$

Theorem 1. Let $\gamma \geqslant 0$. Then, if the function w belongs to the spaces: 1) $E_{\alpha,A}$, $\alpha \geqslant 1$; 2) $E_{-\alpha,A}$, $\alpha \geqslant 1$; 3) $H_{(-k)}$; 4) $H_{(k)}$ for some A, there exists a solution of equation (1) belonging respectively to the spaces: 1) E_{α,A_1}; 2) $E_{-\alpha,A_1}$; 3) $H_{(-k)}$; 4) $H_{(k)}$, for some A_1.

Thus, in the case $\gamma \geqslant 0$, for the existence of a solution it is sufficient that the right-hand side not grow too rapidly. In the case $\gamma < 0$ the situation changes, namely, the faster the growth of the right-hand side, the greater the smoothness that must be required of it in order to ensure the existence of a solution among ordinary functions; for example, if $\gamma = -1$, then for the existence of a solution with arbitrary right-hand side $w \in H_{(-k)}$ it is necessary that the function w be k times differentiable.

However, in a certain sense the converse is also true: if the function w can be expanded in a series $w = \sum_{k=1}^{\infty} w_k$ such that the inequality

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k} C_k^j B^{|j|}|j|^{|j|} a_{k-|j|} \left\|(D+1)^{[-\gamma(j+m)]+1}\frac{w_k}{|x+i|^k}\right\|_{L_2} < \infty, \tag{2}$$

holds, where $[\delta]$ is the integer part of δ, and

$$a_k = \max_x \frac{|x+i|^k}{e(x)},$$

then there exists a solution of equation (1) such that $e^{-1}(x)u(x) \in L_2$.

In order to write condition (2) in a more explicit form for part of the functions w, we introduce spaces of type E:

$$E_{\alpha,A}^{\beta,B} = \left\{\chi(x) :\ |D^q\chi(x)| \leqslant CB^{|q|} \cdot |q|^{|q|/\beta} \exp\left[\frac{1}{A}|x|^{1/\alpha}\right]\right\}$$

The following theorem is proved by using inequality (2).

Theorem 2. Let $\gamma < 0$. Then, if the function w belongs to the spaces: 1) $E_{\alpha,A}^{\beta,B}$ for $\alpha + \gamma\beta \geq 1$, $\alpha > 1$ and some A and B; 2) $S_{\alpha,A}^{\beta,B}$ for $\alpha + \gamma\beta \geq 1$, $\alpha > 1$ and some A and B; 3) $H_{(-k)}$ together with all its derivatives up to order $[-\gamma(k+m)+m+1]$; 4) $H_{(k)}$ with the same number of derivatives, then there exists a solution of

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equation (1) belonging respectively to the spaces: 1) $E_{\alpha,A_1}^{\beta,B_1}$ for some B_1 and A_1; 2) $S_{\alpha,A_1}^{\beta,B_1}$ for some B_1 and A_1; 3) $H_{(-k)}$; 4) $H_{(k)}$.

It is interesting to note the symmetry in the existence classes exhibited by Theorems 1 and 2.

Since all the classes in which we have ensured the existence of a solution are contained in the corresponding uniqueness classes, they are at the same time classes of well-posedness.

We now observe that, excluding the first two classes of Theorem 2, we have ensured the existence of a solution only in the generalized sense; however, in order that it be a solution in the ordinary sense, it is enough to require that not only the function w, but also all its derivatives needed for the operator $p\left(i\frac{\partial}{\partial x}\right)$ to be applicable to it, possess the properties required by these theorems.

We note that the definition of the genus does not require that the polynomial $p(s)$ have no zeros; it is sufficient that it have no zeros outside a bounded domain. We indicate a method for finding the genus of an equation from the values of the polynomial $p(s)$ on a real manifold.

Theorem 3. The genus of equation (1) is equal to the number

$$-\sup \lim_{|\sigma|\to\infty} \left\{ \frac{\ln \dfrac{|\operatorname{grad} p(\sigma)|}{|p(\sigma)|}}{\ln|\sigma|} \right\}; \qquad |\operatorname{grad} p(\sigma)| = \left[\sum_{j=1}^{n} |p'_{s_j}(\sigma)|^2 \right]^{1/2}. \tag{3}$$

In the case when the operator $p\left(i\frac{\partial}{\partial x}\right)$ is hypoelliptic in the sense of Hörmander [4], formula (3) makes it possible to determine the class of Gevrey to which all solutions of the corresponding homogeneous equation belong (see [5]).

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Note: Figure translations are in progress. See original paper for figures.

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