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Abstract

Full Text

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EXTENSION OF MONOTONE MAPPINGS TO MONOTONE MAPPINGS OF BICOMPACTA

(Presented by Academician P. S. Aleksandrov on 11 VI 1960)

At the Second All-Union Topological Conference V. Ponomarev posed the following question: is it always possible to extend a monotone mapping f of a space X onto a space Y to a monotone mapping of some extension* bX onto some extension bY? Here a positive solution of V. Ponomarev' s problem is given for all closed and all open mappings in the case where the extension of the space Y is the Čech (maximal) extension βY. At the end two examples proposed by Yu. M. Smirnov are given. The first shows that even for identity mappings the condition of maximality of the extension bY (i.e. that $bY = \beta Y$) is essential, and the second—that for arbitrary mappings the theorem is false.

Theorem. Let bX be an arbitrary extension of a completely regular space X, let βY be the Čech extension of a normal space Y, and let f_b be a mapping of the bicompactum bX onto βY which is the (unique) extension of a monotone mapping f of the space X onto the space Y; if f is open or closed, then the extension f_b is also monotone.

Proof**. We make the following simple observation.

Remark. Suppose a mapping f of a space X into a space Y is given; if its extension f_β, carrying βX into βY, is monotone, then for every such extension bX of the space X for which the mapping f can be extended to a mapping f_b of the extension bX into βY, this extension f_b is also monotone.

Indeed, under the conditions of the remark, for any such extension bX we have

$$f_\beta = f_b \vartheta,$$

where ϑ is the mapping of the bicompactum βX onto bX that is identical on X. Therefore, if for some point y of βY the preimage*** $f_b^{-1} y$ were disconnected, then the preimage

$$f_\beta^{-1} y = \vartheta^{-1} f_b^{-1} y$$

would also be disconnected.

In view of this remark it is enough to carry out the

Proof of the theorem for the extension f_β. Suppose the hypotheses of the theorem are satisfied and at the same time the preimage $f_\beta^{-1} y$ of some point

$y \in \beta Y$ is disconnected, i.e.

$$f_\beta^{-1} y = A \cup B,$$

where A and B are nonempty, closed, and disjoint. By the normality of the bicompactum βX there exist

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* By an extension we shall everywhere mean only **bicompact** extensions, and by mappings–**continuous** mappings.

** My original proof was somewhat simplified by V. T. Levshenko and Yu. M. Smirnov. I sincerely thank them.

*** Everywhere only **complete** preimages are considered.

there exist neighborhoods U and V of the set A, respectively of the set B, such that $\overline{U}^\beta \cap \overline{V}^\beta = \emptyset$*. Since $A \subseteq \overline{U}^\beta = \overline{U \cap X}^\beta$, it follows that $y \in \overline{f(U \cap X)}^\beta$. Similarly, $y \in \overline{f(V \cap X)}^\beta$. Therefore

$$y \in \overline{f(U \cap X)}^\beta \cap \overline{f(V \cap X)}^\beta. \tag{1}$$

By the normality of the space Y, the closure operator in βY, applied to closed subsets of Y, commutes with the operation of intersection ([1]). Therefore

$$y \in \overline{f(U \cap X) \cap \overline{f(V \cap X)}}^\beta. \tag{2}$$

Since $f_\beta^{-1} y \subset U \cup V$ and the mapping f_β is closed, there exists a neighborhood Oy such that

$$f_\beta^{-1}(Oy) \subseteq U \cup V. \tag{3}$$

First case: $y \in Y$. Then, by connectedness of the set $f^{-1}y$, either $f^{-1}y \subseteq U$, or $f^{-1}y \subseteq V$; suppose, for example, that $f^{-1}y \subseteq U$. Hence, on the basis of (1), we have

$$y \in Oy \cap f(U \cap X) \cap \overline{f(V \cap X)}.$$

If the mapping f is open, then the image $f(U \cap X)$ is open in Y, and therefore the set $H = Oy \cap f(U \cap X) \cap \overline{f(V \cap X)}$ is nonempty. Let $y' \in H$. Then $f^{-1}y' \subseteq f_\beta^{-1}(Oy) \subseteq U \cup V$, $f^{-1}(y') \cap U \neq \emptyset$ and $f^{-1}(y') \cap V \neq \emptyset$, which is impossible. If, however, the mapping f is closed, then there is a neighborhood Uy such that $f^{-1}(Uy) \subseteq Y \setminus \overline{V}^\beta$, since $f^{-1}y \subseteq X \setminus \overline{V}^\beta$. By (1), similarly to the preceding, we obtain that $y \in Uy \cap \overline{f(V \cap X)}$. Therefore the set $H' = Uy \cap f(V \cap X)$ is nonempty, and for any point y' of H' we again obtain a contradiction: $f^{-1}y' \subseteq f^{-1}(Uy) \subseteq Y \setminus \overline{V}^{\beta}$ and $f^{-1}(y') \cap V \neq \emptyset$. Thus, if $y \in Y$, then the preimage $f_{\beta}^{-1}y$ is connected.

Second case: $y \in \beta Y \setminus Y$. By (2), the set $H'' = Oy \cap \overline{\overline{f(U \cap X)} \cap \overline{f(V \cap X)}}$ is nonempty. Let $y' \in H''$. We now show that $f_{\beta}^{-1}(y') \cap \overline{U}^{\beta} \neq \emptyset$. Assuming the contrary, by closedness of the mapping f_{β}, we find a neighborhood Oy' such that $f_{\beta}^{-1}(Oy') \cap \overline{U}^{\beta} = \emptyset$. Then the set $H''' = Oy \cap Oy' \cap \overline{f(U \cap X)}$ would be nonempty, and for any point y'' of H''' we would have $f_{\beta}^{-1}y'' \subseteq f_{\beta}^{-1}(Oy') \subseteq \beta X \setminus \overline{U}^{\beta}$ and $f_{\beta}^{-1}(y'') \cap U \neq \emptyset$, which is impossible. Thus, $f_{\beta}^{-1}(y') \cap \overline{U}^{\beta} \neq \emptyset$. Similarly, $f_{\beta}^{-1}(y') \cap \overline{V}^{\beta} \neq \emptyset$. At the same time, by (3), $f_{\beta}^{-1}y' \subseteq f_{\beta}^{-1}(Oy) \subseteq \overline{U}^{\beta} \cup \overline{V}^{\beta}$. Since $y' \in H'' \subseteq Y$, the preimage $f_{\beta}^{-1}y'$ is connected, which, in view of the three relations just proved, contradicts the condition $\overline{U}^{\beta} \cap \overline{V}^{\beta} = \emptyset$, which the sets U and V satisfy. Thus, the supposition that the preimage of some point $y \in \beta Y$ is disconnected leads, in both possible cases, to a contradiction. The theorem is proved.

Example 1. Let N be a countable space consisting of isolated points, let bN be any countable extension of it (for example, the extension $N \cup \xi$ of P. S. Aleksandrov), let f be the identity mapping of the space N onto itself, and let f_{β} be its natural extension mapping βN onto bN. Since the extension βN is uncountable, there exist points $y \in bN$ with infinite preimages. At all such points the mapping f_{β} is not monotone, since $\dim \beta N = 0$, and hence $\dim f_{\beta}^{-1}y = 0$ for all points y.

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* $\overline{M}^{\beta}$ denotes closure in the Čech extension, $\overline{M}$ denotes closure in the space being extended, and $\emptyset$ denotes the empty set.

Example 2. Let $X = N$, and let Y be an arbitrary countable compactum, and let f be some one-to-one mapping of the space X onto Y. It is discontinuous, not open, and not closed. The extended mapping f_{β} of the extension βX onto $\beta Y = Y$ is nonmonotone by the same considerations as in Example 1.

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References

1. P. S. Aleksandrov, UMN, 2, no. 1 (17) (1946).

Note: Figure translations are in progress. See original paper for figures.