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Abstract

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On the Coincidence of Various Definitions of Dimension for Locally Bicompact Groups

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In this note it is proved that for a locally bicompact group G the equalities $\dim G = \operatorname{ind} G = \operatorname{Ind} G$ hold. It is known that the space of a locally bicompact group G is paracompact. Hence, from Dowker' s theorem on the coincidence, for paracompact spaces, of the dimensions dim and loc dim ([1], p. 108), it follows that $\dim G = \operatorname{loc}\dim G$. But from the fact that the group G locally decomposes into the direct product of a connected local group of Lie (i.e. an open cube) and a zero-dimensional bicompact group ([2], theorem), it is seen that $\operatorname{ind} G = \operatorname{loc}\dim G$, i.e. $\dim G = \operatorname{ind} G$. (The inequality $\dim G \leqslant \operatorname{ind} G$ was derived somewhat earlier by A. Arhangel' skii.) We shall now show that $\operatorname{ind} G = \operatorname{Ind} G$.

Lemma 1. *Suppose we have a mapping f of a bicompactum X onto a bicompactum Y. If the mapping f is locally topological and the sum theorem holds for the dimension* Ind *in X and Y, then* $\operatorname{Ind} X = \operatorname{Ind} Y$.

Proof. For each point $x \in X$ take a neighborhood Ox whose closure is mapped topologically into Y. Then $\operatorname{Ind}[Ox] \leqslant \operatorname{Ind} Y$ and $\operatorname{Ind}[Ox] \leqslant \operatorname{Ind} X$. From the covering $\{Ox\}$ choose a finite covering $\{Ox_i\}$. The images of the sets $[Ox_i]$ cover Y. Hence, by the sum theorem, we obtain $\operatorname{Ind} X \leqslant \operatorname{Ind} Y$ and $\operatorname{Ind} Y \leqslant \operatorname{Ind} X$, as was required to prove.

Definition. Let X be the inverse-limit space of a spectrum $S = \{X_\alpha, \mathfrak{d}_{\beta\alpha}\}$ (for the definition of a spectrum see ([3], p. 36)), and let A be an arbitrary set lying in X. If $A_\alpha = \mathfrak{d}_\alpha A$, then the spectrum $S' = \{A_\alpha, \mathfrak{d}_{\beta\alpha}\}$ for A is called the **natural spectrum** (relative to the spectrum S).

Lemma 2. *Let a bicompactum X be the limit of a spectrum*

$$S = \{X_\alpha, \mathfrak{d}_{\beta\alpha}\},$$

where:

1) *for the dimension* Ind, *the sum theorem holds in any subset X_α;*

2) $\operatorname{Ind} X_\alpha \leqslant r$;

3) *the projections $\mathfrak{d}_{\beta\alpha}$ are locally topological.*

Then

a) *for any bicompactum* $F \subset X$ *there exist arbitrarily close neighborhoods such that, for their boundaries, the natural spectrum satisfies the conditions of the lemma with* r *in condition 2) replaced by* $r-1$*;*

b) $\operatorname{Ind} X \leqslant r$.

Proof. We prove assertion a). Take a bicompactum F and its neighborhood V. Since F is a bicompactum, there exist α_0 and an open set O_{α_0} in X_{α_0} such that

$$F \subset \mathfrak{d}_{\alpha_0}^{-1} O_{\alpha_0} = O \subset [O] \subset V$$

and

$$\operatorname{Ind} \operatorname{Fr} O_{\alpha_0} \leqslant r-1,$$

where Fr denotes the boundary. For all $\beta \geqslant \alpha_0$ consider the sets

$$O_\beta = \mathfrak{d}_{\beta\alpha_0}^{-1} O_{\alpha_0} = \mathfrak{d}_\beta O.$$

It is clear that

$$\mathfrak{d}_{\gamma\beta} \operatorname{Fr} O_\gamma = \operatorname{Fr} O_\beta = \mathfrak{d}_\beta \operatorname{Fr} O.$$

Thus, for the boundary $\operatorname{Fr} O$ we have the spectrum

$$S'' = \{\operatorname{Fr} O_\beta, \mathfrak{d}_{\gamma\beta}\}$$

for $\beta \geqslant \alpha_0$. In this case the $\mathfrak{d}_{\gamma\beta}$ have preserved their local topological character and condition 1) for $\operatorname{Fr} O_\beta$ is also automatically fulfilled; i.e., by Lemma 1 one may con-

include, that $\operatorname{Ind} \operatorname{Fr} O_\beta = \operatorname{Ind} \operatorname{Fr} O_{\alpha_0} \leqslant r-1$. The spectrum S'' is a cofinal part of the natural spectrum $S' = \{\mathfrak{D}_\alpha \operatorname{Fr} O, \mathfrak{D}_{\beta\alpha}\}$ for $\operatorname{Fr} O$. From Lemma 1 we infer that already for every α the inequality $\operatorname{Ind} \mathfrak{D}_\alpha \operatorname{Fr} O \leqslant r-1$ holds, since $\mathfrak{D}_\alpha \operatorname{Fr} O = \mathfrak{D}_{\beta\alpha} \mathfrak{D}_\beta \operatorname{Fr} O$, where $\beta \geqslant \alpha_0$. Conditions 1) and 3) for S' are satisfied automatically. Thus, a) is proved.

We prove b) by induction on $\operatorname{Ind} X_\alpha$. If $\operatorname{Ind} X_\alpha = -1$, then also $\operatorname{Ind} X = -1$. If the lemma is true with r replaced by $r-1$, then b) follows from a).

Lemma 3. *Let the space* X *be locally bicompact and be the limit of a spectrum* $S = \{X_\alpha, \mathfrak{D}_{\beta\alpha}\}$*. For* X_α *and the projections* $\mathfrak{D}_{\beta\alpha}$*, conditions 1), 2), 3) of the preceding lemma are satisfied and, in addition, the condition*

4) $X = \bigcup_{i=1}^{\infty} \Phi_i$, where the Φ_i are bicompacts (i.e. X is finally compact)

and $\operatorname{Ind} \Phi_i \leqslant r$.

Then $\operatorname{Ind} X \leqslant r$.

Remark. If we consider an arbitrary bicompact contained in X, and its natural spectrum, then we shall find ourselves in the conditions of Lemma 2.

Proof of Lemma 3. We prove by induction. For $\operatorname{Ind} X_\alpha = \operatorname{Ind} \Phi_i = -1$ everything is clear.

Take an arbitrary closed set $F \subset X$ and its neighborhood V. For each point $x \in F$ choose a bicompact neighborhood whose closure lies in V. In this cover of F we inscribe a locally finite (in X) countable cover $\{W_i\}$, and in the obtained open cover of F we inscribe a closed cover $\{F_i\}$, where $F_i \subset W_i$.

Consider the natural spectra for the bicompacts $[W_i]$. By Lemma 2, for the bicompacts $F_i \subset W_i$ there exist basic neighborhoods OF_i such that $[OF_i] \subset W_i$, $\operatorname{Ind} \operatorname{Fr} OF_i \leqslant r-1$, and the natural spectra for $\operatorname{Fr} OF_i$ satisfy the conditions of Lemma 2 with r replaced by $r-1$. Denote $\operatorname{Fr} OF_i$ by A_i.

We have

$$\bigcup_i [OF_i] \subseteq \bigcup_i W_i \subset V.$$

Since the system $\{W_i\}$ is locally finite in X, the system $\{A_i\}$ is also locally finite in X, and since the A_i are closed, the set $A = \bigcup_i A_i$ is closed, i.e. $\operatorname{Fr} \bigcup OF_i \subset A$. If we show that $\operatorname{Ind} A \leqslant r-1$, then the same will also be true for $\operatorname{Fr} \bigcup OF_i$.

The set A satisfies condition 4) of the lemma with r replaced by $r-1$ (by construction). The natural spectrum $S' = \{\mathfrak{D}_\alpha A, \mathfrak{D}_{\beta\alpha}\}$ for A automatically satisfies conditions 1) and 3). Let us show that $\operatorname{Ind} \mathfrak{D}_\alpha A \leqslant r-1$. Since $\mathfrak{D}_\alpha A = \bigcup_i \mathfrak{D}_\alpha A_i$, by the sum theorem we obtain $\operatorname{Ind} \mathfrak{D}_\alpha A \leqslant r-1$ (for, by construction, $\operatorname{Ind} \mathfrak{D}_\alpha A_i \leqslant r-1$). Thus, by induction, $\operatorname{Ind} A \leqslant r-1$, i.e. $\operatorname{Ind} \operatorname{Fr} \bigcup OF_i \leqslant r-1$, i.e. $\operatorname{Ind} X \leqslant r$, as was required to prove.

Lemma 4. *For an arbitrary subset A of a Lie group G_α we have $\dim A = \operatorname{ind} A = \operatorname{Ind} A$ (i.e. in an arbitrary subset of a Lie group the sum theorem holds for Ind).*

Proof of Lemma 4. If the group G_α is connected, then it is simply a space with a countable base. If G_α is not connected, then the component of its identity is open-closed in it, i.e. G_α decomposes into a discrete sum of spaces with a countable base, and the assertion of the lemma is obvious.

Lemma 5. *For a projective-Lie group G (for the definition see (2), p. 5), equal to $\bigcup_{n=1}^{\infty} U^n$, where U is a bicompact symmetric neighborhood of the identity, i.e. the space of the group G is finally compact,*

$$\operatorname{ind} G = \operatorname{Ind} G.$$

Proof of Lemma 5. A projective-Lie group G decomposes into a spectrum $S = \{G_\alpha, f_{\beta\alpha}\}$ of Lie groups G_α, where the kernels g_α and $g_{\beta\alpha}$ are homothe morphisms f_α and $f_{\beta\alpha}$ are bicompacta. Since $\operatorname{ind} G = \operatorname{ind} g_\alpha + \operatorname{ind} G_\alpha$ and $\operatorname{ind} G_\beta = \operatorname{ind} g_{\beta\alpha} + \operatorname{ind} G_\alpha$ (see (^{4}), p. 140), it follows that $\operatorname{ind} G_\alpha \leq \operatorname{ind} G_\beta \leq \operatorname{ind} G = r < \infty$, i.e., beginning with some α_0, $\operatorname{ind} G_\beta = \operatorname{ind} G_{\alpha_0}$ for all $\beta \geq \alpha_0$, and hence it follows that $g_{\beta\alpha}$ are zero-dimensional, if the indices are taken $\geq \alpha_0$, as we shall do henceforth. But the bicompacta $g_{\beta\alpha}$, being closed subgroups of Lie groups, are themselves Lie groups, i.e., simply consist of a finite number of points; consequently the projections $f_{\beta\alpha}$ are locally topological. Thus, for the spectrum S condition 3) of Lemma 3 is satisfied. From Lemma 5 for the spectrum S follows the fulfillment of conditions 1) and 2), since $\operatorname{Ind} G_\alpha = \operatorname{ind} G_\alpha \leq \operatorname{ind} G = r$.

Consider the bicompacta $[U]^n$. Their natural spectra satisfy the conditions of Lemma 2, i.e. $\operatorname{Ind}[U]^n \leq r$. Thus condition 4) of Lemma 3 is also satisfied, i.e. $\operatorname{Ind} G \leq \operatorname{ind} G = r$, as was required to prove.

Main theorem. *For every locally bicompact group G,*

$$\dim G = \operatorname{ind} G = \operatorname{Ind} G.$$

Proof. It is known ((^{2}), p. 39) that in every locally bicompact group G there exists an open-closed projective-Lie subgroup G'. Take a bicompact symmetric neighborhood U of the identity lying in G'; then the subgroup

$$G'' = \bigcup_{n=1}^{\infty} U^n$$

of the group G' will satisfy the conditions of Lemma 4 (since a closed subgroup of a projective-Lie group is projective-Lie; see (^{2}), p. 5) and will be open-closed in G', i.e. also in G. The space of the group G therefore decomposes into a discrete sum of spaces homeomorphic to G'', and for G'' all three dimensions coincide, i.e. also $\operatorname{Ind} G = \operatorname{ind} G = \dim G$.

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