MATHEMATICS

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. G. Vilyatser

STABLE GROUPS OF AUTOMORPHISMS

(Presented by Academician A. I. Mal' tsev on 1 XII 1959)

§ 1. In the work ([1]) L. A. Kaluzhnin proved the following theorem:
If a group G has a finite invariant series stable with respect to its group of automorphisms Φ, then the group Φ is nilpotent.

In the case when the Φ-stable series is normal, only the solvability of the group Φ has been proved; the question of its nilpotency remains open for the time being. In the present note we consider some cases in which the group Φ turns out to be nilpotent or locally nilpotent. The main result is the following theorem:

An externally nilpotent group of automorphisms of a group with the maximality condition is nilpotent.

We give some definitions and notation (see also ([2]-[4])). A set of automorphisms Σ of a group G is called stable if in G there is an ascending normal Σ-stable series. If in G there is a local system of Σ-admissible subgroups in which Σ induces stable sets of automorphisms, then Σ is called a locally stable set of automorphisms. If the Σ-stable series is finite, then Σ is called an externally nilpotent set. In particular, the set Σ may consist of only one automorphism, or may be some group of automorphisms of the group G.

Let g and h be elements of the group G; φ and σ elements of its group of automorphisms Φ. We use the usual notation for commutators; analogously, $[g, \varphi(0)] = g$, $[g, \varphi(n)] = [[g, \varphi(n-1)], \varphi]$, and so on. $[G, \Phi]$ is the Φ-commutant of the group G, i.e., the subgroup of the group G generated by all commutators $[g, \varphi]$. By $\hat{g}$ we denote the inner automorphism induced by the element g in the group G. The identity of the group G is denoted by e, that of the group Φ by ε.

§ 2. **Lemma 1.** *Let G be a group; φ and σ automorphisms of this group; $E \subset H \subset G$ a normal series admissible with respect to these automorphisms, with σ inducing identity automorphisms in the factors of this series, and φ in the factor G/H. Let $g \in G$, $[g, \varphi] = h$. Then the following relation holds:*

$$[g, [\sigma, \varphi(n)]] = [[g, \sigma]\varphi^{-1}\hat{h}(n)] \qquad (n = 0, 1, 2, \ldots). \tag{*}$$

Proof. Denote $[\sigma, \varphi(n)] = \sigma_n$, $[g, \sigma_n] = a_n$ $(n = 0, 1, 2, \ldots)$. We have: $[g, \sigma_n] = g^{-1}\sigma_n(g) = a_n$, whence

$$\sigma_n(g) = ga_n.$$

On the other hand,

$$\begin{aligned}\sigma_n(g) &= [\sigma_{n-1}, \varphi](g) = \sigma_{n-1}^{-1}\varphi^{-1}\sigma_{n-1}\varphi(g) = \sigma_{n-1}^{-1}\varphi^{-1}\sigma_{n-1}(gh)\\ &= \sigma_{n-1}^{-1}\varphi^{-1}(ga_{n-1}h) = \sigma_{n-1}^{-1}\varphi^{-1}(gh \cdot h^{-1}a_{n-1}h)\\ &= \sigma_{n-1}^{-1}(g\varphi^{-1}(h^{-1}a_{n-1}h))\\ &= ga_{n-1}^{-1}\varphi^{-1}(h^{-1}a_{n-1}h) = ga_{n-1}^{-1}\varphi^{-1}\hat{h}(a_{n-1}).\end{aligned}$$

Comparing the expressions for $\sigma_n(g)$, we obtain:

$$a_n = a_{n-1}^{-1}\varphi^{-1}\hat{h}(a_{n-1}). \tag{**}$$

If $n = 0$, then $(*)$ is obviously true. Now from $(**)$ it is easy to obtain the validity of $(*)$ for any n.

Lemma 2. *Let H be a locally nilpotent group, φ its nil-automorphism, and $h \in H$. Then $\varphi\hat{h}$ is also a nil-automorphism of the group H.*

Proof. Let $\bar{H} = \{H, \bar{\varphi}\}$ be the cyclic extension of the group H by means of the element $\bar{\varphi}$, inducing in H the automorphism φ. Obviously, $\bar{\varphi}$ will be a nil-element of the group $\bar{H}$, and therefore the group $\bar{H}$ is locally nilpotent ([5]). But then the element $h\bar{\varphi}$ will be a nil-element, and the inner automorphism $\widehat{h\bar{\varphi}}$ a nil-automorphism of the group $\bar{H}$. It remains to note that the automorphism $\widehat{h\bar{\varphi}}$ in the group H induces the automorphism $\varphi\hat{h}$.

Remark. If in the hypothesis of the lemma H is a nilpotent M-group (M-groups are groups with the maximality condition), then $\bar{H}$ is also a nilpotent M-group, whence it follows that the automorphism $\varphi\hat{h}$ has finite nilpotency index, i.e., there exists an $n \geqslant 0$ such that for every $a \in H$ the equality $[a, \varphi\hat{h}(n)] = e$ holds.

Lemma 3. *Let H be a nilpotent group, and let Φ be its outer nilpotent group of automorphisms. Then the group Φ is nilpotent.*

The proof of this lemma is not difficult to obtain by means of a theorem of L. A. Kaluzhnin.

Theorem 1. *Let G be an M-group, and let Φ be its outer nilpotent group of automorphisms. Then the group Φ is nilpotent.*

Proof. In ([3]), B. I. Plotkin proved that, under the hypotheses of the theorem, $[G, \Phi]$ is a nilpotent group. Denote by Σ the invariant subgroup of the group Φ consisting of all automorphisms that induce the identity automorphisms in $[G, \Phi]$. The factor group Φ/Σ is isomorphic to the group of automorphisms of the group $[G, \Phi]$ induced in it by the group Φ, and, by Lemma 3, is nilpotent.

We shall show that Φ is a nil-group. Let φ and σ be arbitrary automorphisms from Φ. Denote $[\sigma, \varphi(i)] = \sigma_i$. In view of the nilpotency of the group Φ/Σ, there exists a $k \geqslant 0$ such that $\sigma_k \in \Sigma$. Let $g \in G$, $[g, \sigma_k] = a$, $[g, \varphi] = h$. By the remark to Lemma 2, there exists an $n \geqslant 0$, independent of the choice of the element g, such that $[a, \varphi^{-1}\hat{h}(n)] = e$, whence, by Lemma 1, we have

$$[g, [\sigma^k, \varphi(n)]] = [g, [\sigma, \varphi(k+n)]] = e.$$

Since the last equality holds identically for all $g \in G$, it follows that $[\sigma, \varphi(k+n)] = e$, i.e. Φ is a nil-group. But an outer nilpotent group of automorphisms of an M-group is itself an M-group ([6]), and therefore the group Φ is nilpotent ([7]).

Corollary. The group of nil-automorphisms of an M-group is nilpotent ([3]).

§ 3. In this section we consider groups of locally stable automorphisms. The following theorem clarifies the role of locally stable elements of a group, i.e., of those elements that induce inner locally stable automorphisms in the group.

Theorem 2. *In an arbitrary group, the radical coincides with the set of all locally stable elements.*

Proof. 1) We shall show that every element of the radical $R(G)$ of the group G is locally stable. Let F be the subgroup generated by an arbitrary finite set of elements from G and by an element $h \in R(G)$. The radical $R(F)$ is a countable locally nilpotent group; therefore in $R(F)$ one can construct an ascending normal series beginning with the cyclic group $\langle h \rangle$ and reaching $R(F)$. It follows that $\langle h \rangle$ is subinvariant in F, i.e. the element h induces a stable automorphism in F.

2) We shall show the converse, i.e. that every locally stable element

$h \in G$ belongs to the radical $R(G)$. Let $[G^{(\alpha)}]$ be a local system of subgroups of the group G, in which $\hat{h}$ induces stable automorphisms,

$$E = G_0^{(\alpha)} \subset G_1^{(\alpha)} \subset ... \subset G_\beta^{(\alpha)} \subset ... \subset G_\gamma^{(\alpha)} = G^{(\alpha)}$$

an $\hat{h}$-stable series of subgroups of $G^{(\alpha)}$. Denote, for all β, $\{G_\beta^{(\alpha)}, h\} = H_\beta^{(\alpha)}$. Then the cyclic subgroup $\langle h \rangle$ will be subinvariant in $H_\gamma^{(\alpha)} = H^{(\alpha)}$. But in this case $\langle h \rangle \subset R(H^{(\alpha)})$, and, since $[H^{(\alpha)}]$ is a local system of the group G, $\langle h \rangle \subset R(G)$ ([8]).

Corollary. If G is an LM-radical group, then every one of its nil-automorphisms φ is locally stable.

Proof. Let $\overline{G} = \{G, \overline{\varphi}\}$ be the cyclic extension of the group G by means of the element $\overline{\varphi}$, inducing in the group G the automorphism φ. In the LM-radical group $\overline{G}$, the element $\overline{\varphi}$ is a nil-element and therefore is contained in the radical $R(\overline{G})$ ([9]). But then, by Theorem 2, $\overline{\varphi}$ is a locally stable element of the group $\overline{G}$ and, evidently, induces in the group G a locally stable automorphism.

Lemma 4. Let Φ be a periodic group of locally stable automorphisms of the group G. Then $[G, \Phi]$ is a periodic locally nilpotent group.

Proof. It follows from ([3]) that, under the conditions of the lemma, $[G, \Phi]$ is a locally nilpotent group. Let P be the periodic part of $[G, \Phi]$ and $\varphi(g) = gh$, where $\varphi \in \Phi$, $g \in G$, $h \in [G, \Phi] \setminus P$. If $\varphi^n = \varepsilon$, then $\varphi^n(g) = g = gh^n a$, where $a \in P$. But the equality $h^n a = e$ is impossible; therefore $[G, \Phi] = P$.

Lemma 5. A finite group of nil-automorphisms of a periodic locally nilpotent group is nilpotent.

Theorem 3. A finite group of locally stable automorphisms Φ of an arbitrary group G is nilpotent.

Proof. We shall show that Φ is a nil-group. Denote by Σ the set of automorphisms inducing identical automorphisms in $[G, \Phi]$. Then Φ/Σ is a nilpotent group (Lemmas 4 and 5). Let $\varphi \in \Phi$, $\sigma \in \Phi$, $[\sigma, \varphi(i)] = \sigma_i$. For some $k \geq 0$ we have $\sigma_k \in \Sigma$. Suppose that $\sigma_i \neq \varepsilon$ for no i. Then, in view of the finiteness of the group Σ, there is an $s > 0$ such that $\sigma_{m+s} = \sigma_m$ $(m \geq k)$, and, since $\sigma_m \neq \varepsilon$, there is a $g \in G$ such that $[g, \sigma_m] = a \neq e$, whence $[g, \sigma_{m+j}] \neq e$ for no $j \geq 0$. Denote $[g, \varphi] = h$. By Lemma 2, $\varphi^{-1}\hat{h}$ is a nil-automorphism; therefore for some $n \geq 0$ we have $[a, \varphi^{-1}\hat{h}(n)] = e$. But then, by Lemma 1, $[g, [\sigma_m, \varphi(n)]] = [g, \sigma_{m+n}] = e$. The contradiction obtained proves the theorem, since a finite nil-group is nilpotent.

Corollary 1. A locally finite group of locally stable automorphisms is locally nilpotent.

Corollary 2. A locally finite group of nil-automorphisms of an LM-radical group is locally nilpotent.

Remark. Corollary 2 refines a result obtained earlier by the author ([4]).

Ural State University
named after A. M. Gorky

Received
23 X 1959

REFERENCES

1. L. **Kaloujnine**, Ber. Math. Tagung, Berlin, 164 (1953).
2. B. I. **Plotkin**, Uspekhi Mat. Nauk, 13, 4, 82 (1958).
3. B. I. **Plotkin**, DAN, 130, No. 5 (1960).
4. V. G. **Vilyatser**, Uspekhi Mat. Nauk, 14, 5 (1959).

5. B. I. **Plotkin**, Uspekhi Mat. Nauk, 9, 3, 181 (1954).

6. D. M. **Smirnov**, Matem. sbornik, 32, 365 (1953).

7. B. I. **Plotkin**, DAN, 107, No. 5, 648 (1956).

8. B. I. **Plotkin**, DAN, 8, No. 3, 341 (1954).

9. B. I. **Plotkin**, Izv. vyssh. uchebn. zaved., matem., No. 1 (1958).

Note: Figure translations are in progress. See original paper for figures.