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**Abstract**

**Full Text**

**V. B. Kudryavtsev**

**A Completeness Theorem for One Class of Automata Without Feedback**

*(Presented by Academician P. S. Aleksandrov, 9 I 1960)*

In note <sup>(3)</sup> the problem was posed of the completeness of elementary automata in several different senses. The completeness problem was completely solved in the first sense, and some facts were reported concerning completeness in the second and third senses. In the present note the question of completeness in the second sense is solved to the end, and other conditions for completeness in the first sense are given; moreover, the whole exposition, unlike <sup>(3)</sup>, is carried out in a purely functional language.

1°. In this section we shall briefly set forth some ideas, due to A. V. Kuznetsov, connected with the notion of a precomplete class.

Let  $R$  be a set of elements of arbitrary nature. In  $R$  there is given a certain collection of operations which, to certain tuples of elements of  $R$ , assigns an element of  $R$ ; it also contains the operation which assigns to a tuple consisting of a single element that very element itself. We shall call this collection of operations a superposition. The set  $[\mathfrak{M}] \subseteq R$  is called the closure of the set  $\mathfrak{M} \subseteq R$  if it contains all and only those elements of  $R$  which are obtained from  $\mathfrak{M}$  by means of the operations of superposition. The set  $\mathfrak{M}$  is called closed if  $[\mathfrak{M}] = \mathfrak{M}$ . In  $R$  select an arbitrary system  $\mathfrak{U}$  of subsets  $U$ .

**Definition.** A set  $K$  is called **complete relative to  $\mathfrak{U}$**  if  $[K] \cap U \neq \Lambda$  for every  $U$  from  $\mathfrak{U}$ .

**Definition.** A set  $N$  is called a **precomplete class relative to  $\mathfrak{U}$**  if: a) it is not complete relative to  $\mathfrak{U}$ ; b) for every element  $a$  not belonging to  $N$ ,  $[a \cup N]$  forms a set complete relative to  $\mathfrak{U}$ . Obviously, a class precomplete relative to  $\mathfrak{U}$  is closed.

Let all precomplete classes  $N$  relative to  $\mathfrak{U}$  be known. Denote them by  $\mathfrak{N}$ . Suppose that any subset  $G \subseteq R$  which is not complete relative to  $\mathfrak{U}$  can be extended, by adjoining elements from  $R$ , to a class precomplete relative to  $\mathfrak{U}$ . Then the following theorem holds:

**Theorem 1.** *A subset  $G$  of elements of  $R$  is complete relative to  $\mathfrak{U}$  if and only if  $G$  is not contained entirely in any one of the precomplete classes of the system  $\mathfrak{N}$ .*

Another approach is also possible. Suppose there is a system  $\mathfrak{N}'$  of sets  $N'$ , each of which is not complete relative to some system  $\mathfrak{U}'$  of subsets  $U'$  of  $R$  and

is not contained in any of the others. Suppose, in addition, that the following condition is fulfilled: a set  $G'$  of elements of  $R$  is complete relative to  $\mathfrak{U}'$  if and only if it is not contained entirely in any one of the sets  $N'$  of the system  $\mathfrak{N}'$ . Then the following assertion is true:

**Theorem 2.** 1) Each set  $N'$  is a precomplete class relative to  $\mathfrak{U}'$ ; 2) in  $R$  there does not exist a precomplete relative to

$U'$  class not belonging to  $\mathfrak{R}$ ; 3) any subset that is not complete with respect to a  $U'$  class can be extended to a precomplete one with respect to a  $U'$  class.

2°. Consider the set  $\tilde{P}_2$  of all pairs  $(f, k)$ , where  $f \in P_2[2]$ ,  $k = 0, 1, \dots$ . The number  $t$  in the pair  $(f, t)$  may be interpreted, for example, as the time required to compute the value of the function  $f$ . In  $\tilde{P}_2$  we introduce, inductively, operations of superposition, which are a concretization of the definition introduced by R. E. Krichevskii <sup>(4)</sup>.

**Definition.** 1) Suppose a pair  $(f(x_1, \dots, x_n), l)$  is given; an arbitrary renaming of the variables in  $f(x_1, \dots, x_n)$  is the result of applying superposition; 2) suppose the pairs

$(f(x_1, \dots, x_n), l), (g_1(y_1, \dots, y_{m_1}), l_1), \dots, (g_s(z_1, \dots, z_{m_s}), l_s), l_1 = \dots = l_s = t,$   
are given. The pair  $(g(u_1, \dots, u_p), r)$ , where  $g(u_1, \dots, u_p) = f(g_{i_1}, \dots, g_{i_n}), r = t + l$ , is the result of applying the operations of superposition.

Such a definition of superposition is justified by a number of examples from the theory of circuits <sup>(3)</sup>. We partition  $\tilde{P}_2$  into a system  $U''$  of subsets  $U''$ , where  $\bigcup U'' = \tilde{P}_2$ ,  $U'' = \{(f_0, k)\}$ , where  $f_0$  is an arbitrary fixed function from  $P_2$ ,  $k = 0, 1, \dots$ . In our case, as  $R$  we take  $\tilde{P}_2$  with the operations of superposition introduced in it. Analogously to 1° we introduce the notions of closure, closed set and, taking as  $U$  the system of subsets  $U''$ , the notions of completeness and of precomplete classes with respect to  $U''$ . Thus, in  $\tilde{P}_2$ , in which the operations of superposition are defined, the problem of completeness with respect to  $U''$  is considered. It is easy to see that the indicated completeness is, in essence, completeness in the second sense of the article <sup>(3)</sup>, meaning that the closure of a complete system  $\{(f, n)\}$  contains all functions from  $P_2$ , while  $n$  runs through some subsequence of the natural numbers. Following Post <sup>(1)</sup>, we introduce a classification of functions.

**Definition.** A function  $f(x_1, \dots, x_n) \in P_2$  is called a function of type  $\alpha$  if  $f(x, \dots, x) \equiv x$ ; a function of type  $\beta$  if  $f(x, \dots, x) \equiv 1$ ; a function of type  $\gamma$  if  $f(x, \dots, x) \equiv 0$ ; a function of type  $\delta$  if  $f(x, \dots, x) \equiv \bar{x}$ .

**Definition.** A function  $f(x_1, \dots, x_n) \in P_2$  is called even if  $f(\alpha_1, \dots, \alpha_n) = f(\bar{\alpha}_1, \dots, \bar{\alpha}_n)$  for every tuple  $(\alpha_1, \dots, \alpha_n)$ .

We shall use the following notation, part of which is borrowed from <sup>(2)</sup>. The set of all functions from  $P_2$  of type  $\alpha$  will be denoted by  $A$ , of type  $\beta$  by  $B$ , of type  $\gamma$  by  $\Gamma$ , of type  $\delta$  by  $\Delta$ , even functions by  $Y$ , linear functions by  $L$ , self-dual functions by  $S$ , monotone functions by  $M$ , functions preserving zero

by  $T_0$ , functions preserving one by  $T_1$ . We describe some closed, incomplete with respect to  $U''$ , sets of elements of  $\widetilde{P}_2$ , and the notation  $f \in D$  that we shall use means that the whole set  $D$  is being considered; the parameter  $q$  occurring below in 10 and 11 takes the values  $0, 1, 2, \dots$

1.  $\widetilde{L} = \{(f, q) : f \in L, q = 0, 1, \dots\}$
2.  $\widetilde{S} = \{(f, q) : f \in S, q = 0, 1, \dots\}$
3.  $\widetilde{M} = \{(f, q) : f \in M, q = 0, 1, \dots\}$
4.  $\widetilde{T}_0 = \{(f, q) : f \in T_0, q = 0, 1, \dots\}$
5.  $\widetilde{T}_1 = \{(f, q) : f \in T_1, q = 0, 1, \dots\}$
6.  $\widetilde{C} = \{(f, q+1), (\varphi, q+1), (\psi, 0) : f \in B, \varphi \in \Gamma, \psi \in A, q = 0, 1, \dots\}$
7.  $\widetilde{E}_0 = \{(f, 0), (\varphi, 0), (0, q+1), (1, q) : f \in B, \varphi \in A, q = 0, 1, \dots\}$
8.  $\widetilde{E}_1 = \{(f, 0), (\varphi, 0), (1, q+1), (0, q) : f \in \Gamma, \varphi \in A, q = 0, 1, \dots\}$
9.  $\widetilde{H} = \{(f, 0), (\varphi, q+1) : f \in S, \varphi \in Y, q = 0, 1, \dots\}$
10.  $\widetilde{W}_r = \{(f, (2q+1)2^r), (0, q), (1, q), (\varphi, (2q+1)2^s), (\psi, 0) : \bar{f} \in M, \varphi \in M, \psi \in M, q = 0, 1, \dots, s = r+1, r+2, \dots\}$
11.  $\widetilde{Z}_r = \{(f, (2q+1)2^r), (\varphi, (2q+1)2^s), (\psi, 0) : f \in \Delta, \varphi \in A, \psi \in A, q = 0, 1, \dots, s = r+1, r+2, \dots\}$

It is not hard to see that none of the indicated sets is contained entirely in another, and the first 5 of them are generated by the precomplete classes from  $P_2^{(2)}$ .

**Theorem 3.** The set of elements  $\mathfrak{F} = \{(f_i, t_i)\}$  is complete with respect to  $\mathfrak{U}''$  if and only if  $\mathfrak{F}$  is not contained entirely in any of the sets:

$$\widetilde{L}, \widetilde{S}, \widetilde{M}, \widetilde{T}_0, \widetilde{T}_1, \widetilde{C}, \widetilde{E}_0, \widetilde{E}_1, \widetilde{H}, \widetilde{W}_r, \widetilde{Z}_r,$$

where  $r = 0, 1, 2, \dots$

Obviously, the clauses of Theorem 2 can now be formulated as corollaries.

**Corollary 1.** Each of the sets indicated in Theorem 3 is a precomplete class with respect to  $\mathfrak{U}''$ .

**Corollary 2.** Apart from the precomplete classes with respect to  $\mathfrak{U}''$  introduced in Theorem 3, there are no others in  $\widetilde{P}_2$ .

**Corollary 3.** Every closed set  $\mathfrak{M}''$  that is not complete with respect to  $\mathfrak{U}''$  is contained in one of the sets of Theorem 3.

**Theorem 4.** From every set complete with respect to  $\mathfrak{U}''$  one can extract a subset complete with respect to  $\mathfrak{U}''$  consisting of no more than 5 elements.

The following example shows that this result cannot be improved.

**Example.** From the complete system

$$\mathfrak{F} = \{(x \& y, 0), (x + y + z, 0), (1, 0), (0, 1), (x, 1)\}$$

one cannot remove a single element without destroying completeness.

The proofs of Theorems 3 and 4 are based on lemmas.

**Lemma 1.** The set  $\mathfrak{F} = \{(\varphi_i, t_i)\}$  is complete with respect to  $\mathfrak{U}'$ , if: a)  $(x, t) \in \mathfrak{F}$ ,  $t > 0$ ; b)  $\{\varphi_i\}$  is a complete system in  $P_2$ ,  $t_i = k_i t$ .

**Lemma 2.** The set  $\mathfrak{F} = \{(\varphi_i, t_i)\}$  is complete with respect to  $\mathfrak{U}''$ , if: a)  $\{\varphi_i\}$  is a complete system in  $P_2$ ; b)  $(x, 2l + 1) \in [\mathfrak{F}]$ .

**Lemma 3.** The set  $\mathfrak{F} = \{(\varphi_i, t_i)\}$  is complete with respect to  $\mathfrak{U}''$ , if: a)  $\{\varphi_i\}$  is a complete system in  $P_2$ ; b)  $(x, t) \in [\mathfrak{F}]$ ,  $(\bar{x}, t) \in [\mathfrak{F}]$ ,  $t > 0$ .

**Lemma 4.** From a function  $f(x, \dots, x_n)$  of type  $\beta$ , not identically one,  $x$  can be obtained with the aid of 1, and  $\bar{x}$  can be obtained with the aid of 0. From a function  $f(x_1, \dots, x_n)$  of type  $\gamma$ , not identically zero,  $x$  can be obtained with the aid of 0, and  $x$  can be obtained with the aid of 1.

**Lemma 5.** Suppose that  $\varphi(x_1, \dots, x_n) \in \bar{L}$ ,  $f(x_1, \dots, x_m) \in \bar{L}$ , and  $\varphi(x_1, \dots, x_n)$  and  $f(x_1, \dots, x_m)$  depend essentially on all their variables; then

$$f(\varphi(x_1, \dots, x_n), \dots, \varphi(x_1, \dots, x_{m-n})) \in \bar{L}.$$

**Lemma 6.** Suppose that  $f(x_1, \dots, x_n) \in \bar{S}$ ,  $\varphi(x_1, \dots, x_m) \in \bar{S}$ , and  $f(x_1, \dots, x_n)$  and  $\varphi(x_1, \dots, x_m)$  depend essentially on all their variables; then

$$f(\varphi(x_1, \dots, x_n), \dots, \varphi(x_1, \dots, x_{m-n})) \in \bar{S}.$$

**Lemma 7.** If  $f(x_1, \dots, x_n)$  is not an even function, then from it one can, with the aid of negation and identification of variables, obtain  $x$ .

3°. For the case of completeness in the first sense<sup>(3)</sup>, one can easily indicate all the corresponding precomplete classes. It turns out that there are 6 in all.

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<sup>3</sup> V. B. Kudryavtsev, *DAN*, **130**, No. 6 (1960).

<sup>4</sup> R. E. Krichevskii, *DAN*, **126**, No. 6 (1959).

*Note: Figure translations are in progress. See original paper for figures.*

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