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J. P. BENZECRI and F. LE ROY (J. P. BENZECRI et F. LE ROY)

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Abstract

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MATHEMATICS

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ON SINGULARITIES OF ANALYTIC FIELDS

(Presented by Academician P. S. Aleksandrov, 5 I 1960)

I. Notation and definitions. Let (F, V, p) be an analytic complex skew product (p denotes the projection onto the base V). An analytic subset S in F is said to define a **holomorphic section over** A (an open subset of V) if p is an isomorphism between A and $p^{-1}(A) \cap S$. It is said that S defines an **analytic over A section** if there exists an open subset A' , everywhere dense in A , over which S defines a holomorphic section. It is said that the analytic section over A defined by means of S is meromorphic if $S \cap p^{-1}(A)$ is a closed subset of $p^{-1}(A)$. A point $a \in A$ is called **regular for the section S** if there exists an open neighborhood O_a of the point a over which S defines a holomorphic section. A nonregular point a is called **singular**. A singular point which has no neighborhood O over which S would define a meromorphic section is called **essentially singular**.

Let (E, V, p) be an analytic complex vector skew product (whose fiber is the vector space C^n); let $P(E)$ be the projective skew product associated with E . We define a 1-field over $A \subset V$ as an analytic section Γ of the skew product $P(E)$ over A . $S(\Gamma)$ is that part of E which is the closure of the preimage of Γ in E .

It is said that a section s of the skew product E is **included in** Γ if $s \subset S(\Gamma)$.

The union of all holomorphic sections included in Γ (defined over some open subset of the space V) is denoted by $I(\Gamma)$.

$\mathcal{O}(V)$ denotes the sheaf of germs of functions holomorphic on V , $\mathcal{O}_x(V)$ the fiber of $\mathcal{O}(V)$ at the point $x \in V$. $\mathcal{F}(I)$ is the sheaf of germs of holomorphic sections of E included in Γ ; $\mathcal{F}(I)$ is a sheaf of modules over the sheaf of rings $\mathcal{O}(V)$; $\mathcal{F}_x(I)$ is the fiber of $\mathcal{F}(I)$ over x . f_x (respectively s_x) denotes the germ at x of the function f (respectively of the section s).

Since the results obtained by us follow from a local study, we shall assume that V is an open subset of C^p and that $E = V \times C^n$ ($p \geq 2$, $n \geq 2$).

A section s of the skew product E is defined by means of n analytic functions X_1, \dots, X_n .

II. Main results.

Proposition 1. *Let s be a holomorphic section of E over A , not identically equal to zero. Then there exists one and only one meromorphic over A 1-field Γ such that $s \in S(\Gamma)$.*

Proposition 2. *Let Γ be a meromorphic over A 1-field and let $a \in A$. Then there exists such a neighborhood O_a of the point a that over O_a a holomorphic section s in E is defined with the following properties:*

- a) $s \in S(\Gamma)$;
- b) Let $s_x \in \mathcal{F}(I)$ and $x \in A$. Then there exists $\gamma_x \in \mathcal{O}_x(V)$ such that $s_x = \gamma_x \tilde{s}_x$;
- c) the set of zeros of s has dimension $\leq p - 2$.

Proposition 3. The existence and commutativity of the following diagram are proved:

$$\mathcal{F}(I) \begin{matrix} \xrightarrow{\mathfrak{D}^{-1}} \\ \xleftrightarrow{\mathfrak{D}} \\ \xrightarrow{\mathfrak{D}} \end{matrix} \mathcal{F}(F)$$

$$\begin{array}{ccc} \mathcal{F}(I) & & \mathcal{F}(F) \\ \downarrow b_1 & \searrow q & \downarrow b_F \\ I & \xleftarrow{D} & F \end{array}$$

- a) b_1 is the mapping that transforms the germ s_x at x of a section s into $s(x)$;
- b) F is the quotient sheaf of $\mathcal{F}(I)$ by the following equivalence relation: $q(s_x) = q(\tilde{s}_x)$, if $s_x = g_x \tilde{s}_x$ and $g(x) = 1$ ($g_x \in \mathcal{O}_x(V)$).

It turns out that F has the structure of a vector analytic oblique product over V with fiber C .

- c) D is completely determined by the relation $D \circ q = b_1$;
- d) $\mathcal{F}(F)$ is the sheaf of germs of holomorphic sections of the oblique product F ;
- e) \mathfrak{D} is the mapping of the sheaf $\mathcal{F}(I)$ into $\mathcal{F}(F)$ induced by the mapping D .

It turns out that \mathfrak{D} is an isomorphism between the two sheaves and that the inverse mapping \mathfrak{D}^{-1} satisfies the relation $b_F \circ \mathfrak{D}^{-1} = q$ (b_F is the mapping of $\mathcal{F}(F)$ into F).

Proposition 4. Let Γ be an analytic 1-field over V . In order that Γ be meromorphic, it is necessary and sufficient that the set of singularities of the field have dimension $\leq p - 2$.

Suppose now that $p = n$ and that Γ has an isolated singularity at 0.

Definition 1. Let Σ be an oriented sphere of dimension $2n - 1$, whose linking index with $\{0\}$ is equal to 1. Then the 1-field Γ defines a mapping of Σ into $P(n - 1, C)$. The class of this mapping in $\pi_{2n-1}(P(n - 1, C)) \simeq Z$ is an integer called the **index of the singularity** of Γ at 0.

Proposition 5. Let s be a section in E , defined by Proposition 2; the index of the singularity at 0 is equal to the index of the mapping s of the base V into the fiber C^n (over the zero of the fiber).

Proposition 6. Let r be the projection of $C^n - \{0\}$ onto its quotient space $S(2n - 1)$. Then $r \circ s$ defines a mapping of the sphere Σ into $S(2n - 1)$, whose class in $\pi_{2n-1}(S(2n - 1)) \simeq Z$ is equal to the index of the singularity.

It turns out that if E is regarded as a real $2n$ -dimensional vector oblique product, then the index of Γ at 0 coincides with the Kronecker index of s .

Proposition 7. The index is an integer positive number, which can take all values different from 0.

Example. According to Proposition 1, the field is completely determined by a section s in E . Let us take as s the section defined by the following functions:

$$X_1 = x_1^p; \quad X_i = x_i \quad (i = 1, \dots, n)$$

(where x_i are the coordinates in V , and X_i are the coordinates of the fiber). Then the index of the singularity is equal to p .

III. **Proof scheme.** The main point is Proposition 2. It is proved as follows. Suppose that X_1 is not identically equal to 0 over $I(\Gamma)$. Then the fractions $\varphi_i = X_i/X_1$ ($i = 2, \dots, n$), computed over $I(\Gamma)$, are meromorphic functions defining the field Γ .

Denote by $D_i^+ - D_i^-$ the divisor of the function φ_i in a (where D_i^+, D_i^- are positive divisors without common divisors). Denote by D_1' the sum $\sum D_i^-$ ($i = 2, \dots, n$); by D_i' ($i = 2, \dots, n$) the expressions $D_i^+ + D_1' - D_i^-$; and by D the greatest common divisor of all D_i' ($i = 1, \dots, n$).

Let φ be a holomorphic function, defined up to a holomorphic factor different from 0 in a , whose divisor is equal to $D_1' - D$. Then $X_1 = \varphi$, $X_i = \varphi \times \varphi_i$ ($i = 2, \dots, n$) are n holomorphic functions that define the section s^* .

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* Definitions of the concepts occurring in the present article may be found in works (1-3).

Note: Figure translations are in progress. See original paper for figures.

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