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Abstract

Full Text

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ON THE THEORY OF QUASICONFORMAL MAPPINGS IN SPACE

(Presented by Academician M. A. Lavrent'ev on 17 II 1960)

Here we consider applications of the method of moduli, extended to space ⁽¹⁾. This method leads to simple proofs of the theorems of M. A. Lavrent'ev ⁽²⁾ and M. A. Kreines ⁽³⁾ on quasiconformal mappings in space, as well as to a certain strengthening of them.

By a **quasiconformal mapping** of a spatial domain D we shall here mean a homeomorphic mapping $P_* = f(P)$ of this domain, possessing at each of its points continuous partial derivatives and a positive Jacobian J . At each point $P \in D$, the principal linear part of f transforms spheres into similar and similarly situated ellipsoids, which we shall call **characteristic**; the ratios of the semiaxes $p = a/c$ and $q = b/c$ ($a \geq b \geq c$) of these ellipsoids will be called the **characteristics** of the mapping at the point P . Quasiconformal mappings with characteristics bounded by a constant Q are called **Q -quasiconformal**.

1°. We begin with a lemma which extends to space the well-known lemma of Grötzsch-Teichmüller (see, for example, ⁽⁴⁾).

Lemma. *Among all families of smooth homeomorphic sphere surfaces lying in the ball $OP < 1$ and enclosing continua γ that contain O and a fixed point $P_0 \neq O$, the greatest modulus is attained by the family of surfaces $\{S\}$ enclosing the rectilinear segment OP_0 .*

Let r, φ, z be cylindrical coordinates (with origin O , the z -axis along OP_0); note first of all that, in computing the modulus of the extremal family $\{S\}$, one may restrict oneself to metrics of the form $\rho = \rho(r, z)$. Indeed, along with every admissible $\rho(r, \varphi, z)$, the metrics $\rho_\alpha(r, \varphi, z) = \rho(r, \varphi + \alpha, z)$ ($0 < \alpha < 2\pi$) are also admissible, for together with S the family also contains the surfaces obtained from S by rotations about the z -axis. But then the metric

$$\rho_0(r, z) = \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} \rho_\alpha^2 d\alpha \right)^{1/2} = \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} \rho^2 d\varphi \right)^{1/2}$$

is also admissible, since

$$\int_S \rho_0^2 d\sigma = \frac{1}{2\pi} \int_S d\sigma \int_0^{2\pi} \rho_\alpha^2 d\alpha = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \int_S \rho_\alpha^2 d\sigma \geq \frac{1}{2\pi} \int_0^{2\pi} d\alpha = 1$$

for every S . But, by Hölder's inequality, the volume of the ball in the metric ρ_0 ,

$$\int_V \rho_0^3 d\omega = \frac{1}{2\pi\sqrt{2\pi}} \int_V d\omega \left(\int_0^{2\pi} \rho_\alpha^2 d\alpha \right)^{3/2} \leq \frac{1}{2\pi} \int_V d\omega \int_0^{2\pi} \rho_\alpha^3 d\alpha = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \int_V \rho_\alpha^3 d\omega = \int_V \rho^3 d\omega$$

does not exceed the volume of this ball in the metric ρ .

It is also not difficult to see that $M\{S\} = M\{\tilde{S}\}$, where $\{\tilde{S}\}$ is the family of all smooth surfaces, homeomorphic to a sphere, of revolution about the z -axis and enclosing OP_0 .

Now let $\{\Sigma\}$ be a family of smooth surfaces, homeomorphic to a sphere, enclosing the continuum γ . Denote by $\tilde{\Sigma}$ the surface obtained from Σ by the following symmetrization process: in each plane $z = z_0$ one constructs a circle with center on the z -axis whose area πr_0^2 is equal to the area of the section, by this plane, of the domain bounded by Σ ; $\tilde{\Sigma}$ is formed by the circumferences of such circles. The family $\{\Sigma\}$ passes into the family of all surfaces of revolution $\{\tilde{\Sigma}\}$ enclosing the continuum $\tilde{\gamma}$. Since $\tilde{\gamma} \supset OP_0$, it follows, by Theorem 4 of ⁽¹⁾, that

$$M\{\tilde{\Sigma}\} \leq M\{\tilde{S}\} = M\{S\}.$$

Denote by $\{\Sigma_t\}$ an arbitrary one-parameter family of surfaces from $\{\Sigma\}$ simply covering the ball with the continuum γ removed; let $\{\tilde{\Sigma}_t\}$ be the family obtained by its symmetrization; it simply covers the ball with $\tilde{\gamma}$ removed. Let $\tilde{\rho}(r, z)$ be an arbitrary admissible metric for $\{\tilde{\Sigma}_t\}$; define the metric ρ by setting it equal to $\tilde{\rho}(r_0, z_0)$ at all points of the section of Σ_t by the plane $z = z_0$. The metric ρ is admissible for $\{\Sigma_t\}$, since symmetrization does not increase surfaces, and, since symmetrization does not change volumes,

$$\int_{V \setminus \gamma} \rho^3 d\omega = \int_{V \setminus \tilde{\gamma}} \tilde{\rho}^3 d\omega.$$

Hence it follows that

$$M\{\Sigma_t\} \leq M\{\tilde{\Sigma}_t\} \leq M\{\tilde{\Sigma}\} \leq M\{S\},$$

and from this it is not hard to derive also the inequality $M\{\Sigma\} \leq M\{S\}$. The lemma is proved.

By invariance under rotations, the modulus of the extremal family depends only on $r = OP_0$; we shall denote it by $\nu = \nu(r)$. The function $\nu(r)$ is decreasing,

$\nu(0) = \infty, \nu(1) = 0$. Using the Grötzsch principles from ⁽¹⁾ and supplementary conformal self-mappings of the ball, one can obtain for it the following estimates:

$$\frac{1}{2\sqrt{\pi}} \ln \frac{1}{r} < \nu(r) < \frac{1}{2\sqrt{\pi}} \ln \frac{16}{r}. \quad (1)$$

2°. The following theorem extends to space the well-known theorem of M. A. Lavrent'ev on distortion ⁽⁵⁾, in the form given to it by Hersch ⁽⁶⁾.

Theorem 1. Let $P_* = f(P), f(O) = O$, be an arbitrary Q -quasiconformal mapping of the ball $\overline{OP} < 1$ onto itself; for any point of the ball we have

$$\frac{1}{Q} \nu(\overline{OP}) \leq \nu(\overline{OP}_*) \leq Q \nu(\overline{OP}). \quad (2)$$

The proof is based on the lemma of 1° and Theorem 1 of ⁽¹⁾. Using the monotonicity of $\nu(r)$, one may rewrite (2) in the form

$$\nu^{-1}\{Q\nu(\overline{OP})\} \leq \overline{OP}_* \leq \nu^{-1}\left\{\frac{1}{Q}\nu(\overline{OP})\right\}. \quad (3)$$

Corollary 1 (M. Ried ⁽⁷⁾). A conformal mapping of the ball $\overline{OP} < 1$ onto itself, normalized by $f(O) = O$, reduces to a rotation.

This is obtained from (3) for $Q = 1$.

Corollary 2. Under the hypotheses of Theorem 1,

$$\left(\frac{\overline{OP}}{16}\right)^Q \leq \overline{OP}_* \leq 16(\overline{OP})^{1/Q}. \quad (4)$$

This estimate substantially sharpens the estimate of M. A. Kreines ⁽³⁾; as the example of the mapping $r_* = r^{1/Q}, \theta_* = \theta, \varphi_* = \varphi$ (r, φ, θ are polar coordinates) shows, its order is sharp.

Remark. If one applies a slightly modified method of continuation by symmetry ⁽⁵⁾, then estimate (4) can be extended to arbitrary points P_1 and P_2 of the ball $\overline{OP} \leq 1$:

$$(\overline{P_1P_2}/K)^Q \leq \overline{P_{1*}P_{2*}} \leq K(\overline{P_1P_2})^{1/Q}, \quad (5)$$

where $K \leq 17 \cdot 32^Q$.

3°. The method under consideration leads to a simple proof of a theorem of M. A. Lavrent'ev (2); extending the method of K. Andreian-Cazacu (8) to space makes it possible to strengthen the result. Suppose that in D there is given a family $\{S\}$ of smooth surfaces such that through every point $P \in D$ there passes

one and only one S , and a quasiconformal mapping $P_* = f(P)$. Let $\alpha_1, \alpha_2, \alpha_3$ denote the angles formed by the normal to S at P with the semiaxes of the characteristic ellipsoid. The element of area of S cut out by this ellipsoid is equal to

$$d\sigma = \frac{\pi abc}{\sqrt{a^2 \cos^2 \alpha_1 + b^2 \cos^2 \alpha_2 + c^2 \cos^2 \alpha_3}}.$$

Let r be the radius of the sphere corresponding to the ellipsoid; then the corresponding area element is $d\sigma_* = \pi r^2$, and the Jacobian is $J = r^3/abc$; therefore

$$\frac{d\sigma_*}{d\sigma} = J^{2/3} \left\{ \left(\frac{p^2}{q} \right)^{2/3} \cos^2 \alpha_1 + \left(\frac{q^2}{p} \right)^{2/3} \cos^2 \alpha_2 + \left(\frac{1}{pq} \right)^{2/3} \cos^2 \alpha_3 \right\}^{1/2} = J^{2/3} \{m(P)\}^{2/3}. \quad (6)$$

Let $\rho(P)$ be an admissible metric for $\{S\}$; for the corresponding family $\{S_*\}$ put

$$\rho_*(P_*) = J^{-1/3} \{m(P)\}^{-1/3} \rho(P).$$

Since by (6) $\rho_*^2 d\sigma_* = \rho^2 d\sigma$, this metric is admissible for $\{S_*\}$. The volume of D_* in this metric is

$$\int_{D_*} \rho_*^3 d\omega_* = \int_D \frac{\rho^3(P)}{m(P)} d\omega.$$

Suppose further that the surfaces $S = S_t$ depend on a parameter t , $t_1 < t < t_2$, with $d\omega = d\sigma_t dt$ ($d\sigma_t$ is the area element of S_t), and denote

$$m(t) = \max_{P \in S_t} m(P).$$

Using Hölder's inequality, we find

$$\int_{D_*} \rho_*^3 d\omega_* \geq \int_{t_1}^{t_2} \frac{dt}{m(t)} \int_{S_t} \rho^3 d\sigma_t \geq \int_{t_1}^{t_2} \frac{dt}{m(t)} \frac{1}{\sqrt{\sigma(t)}} \left(\int_{S_t} \rho^2 d\sigma_t \right)^{3/2},$$

where $\sigma(t)$ is the area of S_t . Since ρ is admissible for $\{S_t\}$, finally

$$\int_{D_*} \rho_*^3 d\omega_* \geq \int_{t_1}^{t_2} \frac{dt}{m(t) \sqrt{\sigma(t)}}. \quad (7)$$

On the basis of these considerations one proves

Theorem 2. Let $P_* = f(P)$ be a quasiconformal mapping of the unit ball with the center removed onto itself, mapping the unit sphere onto itself. Denote by α_i the angles formed by OP with the semiaxes of the characteristic ellipsoid and

$$m(r) = \max_{OP=r} m(P), \quad 0 < r < 1.$$

If

$$\int_0^1 \frac{dr}{r m(r)} = \infty, \quad (8)$$

then there exists $\lim_{P \rightarrow O} f(P)$.

Indeed, suppose that the set of limit values of $f(P)$ at the point O is a continuum γ , not a point; by subjecting the ball to an additional conformal mapping, one can arrange that γ contains O .

By the lemma from 1°, the modulus of the family of surfaces surrounding γ is finite. On the other hand, for any admissible metric ρ_* , according to (7),

$$\int_{D_*} \rho_*^3 d\omega_* \geq \frac{1}{2\sqrt{\pi}} \int_0^1 \frac{dr}{r m(r)} = \infty.$$

Remark. Let

$$p(r) = \max_{OP=r} p(P);$$

obviously, $m(r) \leq p(r)$, and condition (8) is satisfied if

$$\int_0^1 \frac{dr}{r p(r)} = \infty.$$

This condition is certainly satisfied for Q -quasiconformal mappings.

4°. The following two theorems are special cases of a theorem of M. A. Lavrent'ev (2).

Theorem 3. There does not exist a Q -quasiconformal mapping of the half-space $z > 0$ with the removed segment $l = \{x = y = 0, 0 < z \leq H\}$ onto the half-space $z_* > 0$ (x, y, z are Cartesian coordinates).

If l is transformed into a point, let it be O , i.e., suppose that there exists $\lim_{P \rightarrow l} f(P) = O$ as $P \rightarrow l$. Then we consider a cylinder of radius R and height H , having l as its axis, and cover it by a hemisphere of the same radius with center at the upper end of l . Let $\{S\}$ be the family of all quadrable surfaces lying

inside this surface and surrounding l , whose boundaries lie on the lower base of the cylinder; let S_1 and S_2 be the parts of these surfaces lying respectively in the cylinder and in the hemisphere. By Theorem 5 and formulas (10) and (13) of ⁽¹⁾, the moduli of these families are connected by the relation

$$\frac{1}{M^2\{S\}} \geq \frac{1}{M^2\{S_1\}} + \frac{1}{M^2\{S_2\}} = \frac{\pi H}{2R},$$

and, consequently, $M\{S\}$ is finite. At the same time, for the corresponding family $M\{S_*\} = \infty$; the contradiction excludes this case.

If l is not transformed into a point, then the set of indeterminacy for l , i.e., the totality of all limit values $f(P)$ over all possible sequences tending to points of l , is a continuum distinct from a point and lying in the plane $z_* = 0$. From topological considerations it is clear that if, for some point $P_0 \in l$, the set of indeterminacy degenerates into a point P_* , then for the entire segment from P_0 to the upper end of l this set coincides with P_* . Therefore, in the case under consideration there exists a segment $OP_0 \subset l$ such that, for all its points, the set of indeterminacy is a continuum distinct from a point. Consider a cylinder of radius R and height $h = OP_0$, with the axis OP_0 removed, and in it the family of all closed rectifiable curves surrounding the axis. By formula (12) of ⁽¹⁾, the modulus of this family is infinite, but corresponding to it is a family of curves whose lengths are bounded below by a positive constant and which therefore has finite modulus. The theorem is proved.

By analogous methods one proves

Theorem 4. *There does not exist a Q -quasiconformal mapping of the half-space $z > 0$ with the removed piece of the plane $\Pi = \{x = 0, -R \leq y \leq R, 0 < z \leq H\}$ onto the half-space $z_* > 0$.*

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Note: Figure translations are in progress. See original paper for figures.

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