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Abstract

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MATHEMATICS

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ON TWO ANALYTIC METHODS IN THE THEORY OF SMALL SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS

(Presented by Academician A. N. Kolmogorov on 7 IV 1960)

In the theory of nonlinear integral equations, various methods are used to study the branching of solutions caused by changes in parameters: analytic ([1−6]), variational, and topological ([7]). Two analytic methods, first developed by A. M. Lyapunov ([1]) and A. I. Nekrasov ([3]), are usually considered independently of one another, and the study of each of them is carried out by means of special techniques. Thus, for example, in studies devoted to the development of Nekrasov' s method, one usually seeks sufficient conditions for the formal solvability of the equations to which the application of the method is reduced, and constructs convergent numerical majorant series for the functional series that formally satisfy the equation.

In the papers ([6−9]) the author (at the suggestion of M. A. Krasnosel' skii) undertook an attempt to study both analytic methods jointly. It turned out that the application of Lyapunov' s method leads to essentially new theorems on the convergence of Nekrasov' s method.

The implementation of Nekrasov' s method requires solving an infinite system of algebraic equations (see below, system (11)). The general character of the solutions (their number, the nature of their dependence on the parameters) is determined by what the solutions of this system are. In the Lyapunov–Schmidt method, the general character of the solution of the integral equation is determined by the solutions of the so-called branching equation (see below, equation (7)). Since both the Lyapunov–Schmidt method and the Nekrasov–Nazarov method lead to the same solutions, it is natural to suppose that the system of equations appearing in the Nekrasov–Nazarov method is, in some sense, equivalent to the Lyapunov–Schmidt branching equation. In the present note it is shown that the system from the Nekrasov–Nazarov method is not only equivalent to the Lyapunov–Schmidt branching equation, but in a natural sense coincides with it (it coincides if the Lyapunov–Schmidt branching equation is expanded in a series with respect to the corresponding parameter and the condition that the function be equal to zero is replaced by the condition that all coefficients of the expansion vanish).

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1. Consider the integral equation

$$\varphi(x) = \int_0^1 A_{10}(x,y)\varphi(y)\,dy + \int_0^1 \Gamma[x,y,\varphi(y);\lambda]\,dy, \tag{1}$$

where

$$\Gamma(x,y,z;\lambda) = \lambda A_{01}(x,y) + \sum_{k+l\geq 2} A_{kl}(x,y) z^k \lambda^l. \tag{2}$$

For simplicity we shall assume that the functions $A_{kl}(x,y)$ $(k,l = 0,1,2,...)$ are continuous in the square $0 \leq x,y \leq 1$ and that $|A_{kl}(x,y)| \leq A$. The numerical parameter λ may be regarded as real or complex.

If 1 is not an eigenvalue of the kernel $A_{10}(x,y)$, then for small λ equation (1) has a unique small solution, which can be found either by the method of successive approximations or in the form of a series in integral powers of λ.

Difficulties arise if 1 is an eigenvalue of the kernel $A_{10}(x,y)$. In this case, for small λ, equation (1) may have several small solutions representable in the form of series in integral or fractional powers of λ.

We restrict ourselves to the case where 1 is a simple eigenvalue of the kernel $A_{10}(x,y)$; denote by $\omega(x)$ the corresponding eigenfunction of the kernel. The eigenfunction of the transposed kernel $A_{10}(y,x)$ corresponding to the eigenvalue 1 will be denoted by $\omega^*(x)$. Below, $L(x,y)$ will denote the kernel

$$L(x,y) = A_{10}(x,y) - \omega(y)\omega^*(x); \tag{3}$$

this kernel does not have 1 as an eigenvalue; denote its resolvent by $R(x,y)$.

2. The Lyapunov–Schmidt method ([1,2]) for constructing small solutions of equation (1) consists in seeking the desired solution $\varphi(x,\lambda)$ in the form of the series

$$\varphi(x,\lambda) = \psi(x,\lambda) + \alpha(\lambda)\omega(x), \tag{4}$$

where

$$\int_0^1 \psi(x,\lambda)\omega(x)\,dx = 0. \tag{5}$$

The function $\psi(x,\lambda)$ in (4) is a solution of the equation

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$$\psi(x,\lambda) = \int_0^1 L(x,y)\psi(y,\lambda)\,dy + \int_0^1 \Gamma[x,y,\psi(y,\lambda)+\alpha(\lambda)\omega(y);\lambda]\,dy, \tag{6}$$

which, for small α, has a unique solution that is a solution of equation (1), if

$$\int_0^1 \left\{ \int_0^1 \Gamma[x,y,\psi(y,\lambda)+\alpha(\lambda)\omega(y);\lambda]\,dy \right\} \omega^*(x)\,dx = 0. \tag{7}$$

The left-hand side $F(\alpha,\lambda)$ of this equation has the form

$$F(\alpha,\lambda) = \sum_{k+l\geq 1} T_{kl}\alpha^k\lambda^l. \tag{8}$$

To every small solution $\alpha = \alpha(\lambda)$ of equation (7) there corresponds a solution $\varphi(x,\lambda)$ of equation (1), determined by formula (4).

Equation (7) is called the **Lyapunov–Schmidt branching equation**.

In the actual determination of the function $\alpha(\lambda)$, it is usually sought in the form of a series

$$\alpha = \sum_{k\geq 1} \alpha_k \lambda^{k/s}, \tag{9}$$

where s is some number, $s \geq 1$. Without loss of generality one may assume that $s = 1$ (since otherwise the parameter λ could be replaced by the parameter $\mu = \lambda^{1/s}$, i.e., one should seek α in the form of a series

$$\alpha = \sum_{k\geq 1} \alpha_k \lambda^k. \tag{10}$$

The branching equation (7) is thereby transformed into the system

$$f_i(\alpha_1, \alpha_2, \dots, \alpha_i) = 0 \quad (i = 1, 2, \dots) \tag{11}$$

for determining the coefficients of the series (10). The left-hand sides of equation (11) are the coefficients in the expansion of the left-hand side of equation (7) in powers of λ. The infinite system (11) is completely equivalent to specifying equation (7); therefore we shall also call it the Lyapunov–Schmidt branching equation.

Thus, the application of the Lyapunov–Schmidt method requires, on the one hand, knowledge of the resolvent $R(x,y)$ and, on the other, the solution of the system (11).

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3. The Nekrasov–Nazarov method for constructing small solutions of equation (1) ([3,4]) consists in seeking the desired solution $\varphi(x, \lambda)$ in the form of the series

$$\varphi(x) = \sum_{k\geq 1} \varphi_k(x)\lambda^k \tag{12}$$

(or in the form of series in powers of $\mu = \lambda^{1/s}$; this latter case, as we noted in the preceding paragraph, requires no special consideration). The series (12) is substituted into equation (1), the right-hand side is expanded in powers of λ, and then the coefficients of equal powers of λ are compared. After this, from the resulting system of equations

$$\varphi_k(x) = \int_0^1 A_{10}(x, y)\varphi_k(y)\, dy + B_k(x, \varphi_1, \dots, \varphi_{k-1}) \tag{13}$$

the functions $\varphi_k(x)$ are determined. In equations (13) the functions $B_k(x, \varphi_1, \dots, \varphi_{k-1})$ can be written explicitly.

For equations (13) to be solvable it is necessary and sufficient that the conditions

$$\int_0^1 B_k(x)\omega^*(x)\, dx = 0 \quad (k = 1, 2, \dots). \tag{14}$$

be satisfied.

When these conditions are fulfilled, the functions $\varphi_k(x)$ are determined nonuniquely—they can be represented in the form

$$\varphi_k(x) = \psi_k(x) + \alpha_k\omega(x), \tag{15}$$

where

$$\int_0^1 \psi_k(x)\omega(x)\, dx = 0.$$

Then $\psi_k(x)$ may be regarded as the unique solution of the equation

$$\psi_k(x) = \int_0^1 L(x, y)\psi_k(y)\, dy + B_k(x, \varphi_1, \dots, \varphi_{k-1}). \tag{16}$$

The problem of constructing the solution (12) is then reduced to finding such α_k as satisfy the system

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$$g_i(\alpha_1, \dots, \alpha_i) = 0, \tag{17}$$

where

$$g_i(\alpha_1, \dots, \alpha_i) = \int_0^1 B_{i+1}[x, \psi_1(x) + \alpha_1\omega(x), \dots, \psi_i(x) + \alpha_i\omega(x)]\omega^*(x)\,dx.$$

It is natural to call the system (17) the Nekrasov—Nazarov branching equation.

The preceding arguments show that the application of the Nekrasov-Nazarov method requires, on the one hand, the solution of the branching equation (17) and, on the other hand, knowledge of the resolvent $R(x, y)$ (for solving equations (16)).

4. The considerations set forth in the preceding sections show that the applications of the Lyapunov-Schmidt method and the Nekrasov-Nazarov method require overcoming the same technical difficulties: the construction of the resolvent $R(x, y)$ and the solution of the branching equation (11) or (17).

Theorem. *The Lyapunov-Schmidt branching equation (11) coincides with the Nekrasov-Nazarov branching equation (17).*

The Lyapunov-Schmidt theory has been developed in detail (see, for example, ([1,2,5])). Therefore, many results concerning the Nekrasov-Nazarov method follow directly from the theorem stated above. This applies both to the fundamental works ([3,4]) and to numerous later works. In particular, the result obtained by the author ([6]) (namely, that every series of the form (12) formally satisfying equation (1) converges uniformly in some circle $|\lambda| < \sigma$) can also be regarded as a consequence of this theorem.

Let us note that the theorem formulated admits, without special difficulty, an extension to nonlinear operator equations of general form with analytic nonlinearities. On the other hand, this theorem extends to the case when 1 is a multiple eigenvalue of the kernel $A_{10}(x, y)$.

The proof of the theorem stated is obtained by direct calculation. It is easy to see that both the functions $f_i(a_1, \dots, a_i)$ and the functions $g_i(a_1, \dots, a_i)$ (the left-hand sides of equations (11) and (17)) are polynomials in $a_1, \dots, a_i$. Recurrence formulas are derived for constructing these polynomials. It turns out that the recurrence formulas are identical. The coincidence of the polynomials of lower orders is verified directly. It should be noted that, from the technical point of view, the computations carried out are rather cumbersome.

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