ON THE STRUCTURAL ISOMORPHISM OF MODULES OVER REGULAR RINGS

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

L. A. SKORNYAKOV

ON THE STRUCTURAL ISOMORPHISM OF MODULES OVER REGULAR RINGS

(Presented by Academician P. S. Aleksandrov, 8 XII 1959)

Let F^n be a free unitary module with n generators over a regular ring F*. The submodules of the module F^n possessing a finite number of generators form a Dedekind structure $\mathfrak{C}(F^n)$ with complements ((3), p. 146, Theorem 3.2, p. 184, Theorem 2.1; p. 186, Theorem 3.2). A natural question arises concerning the connection between the modules F^n and G^m if the structures $\mathfrak{C}(F^n)$ and $\mathfrak{C}(G^m)$ are isomorphic. If $n \geq 3$, and F and G are fields, then a structural isomorphism is induced by a semilinear mapping of the module F^n onto the module G^m ((2), p. 62). If the regular ring F has a system of idempotents $\varepsilon_1, \dots, \varepsilon_n$, $n \geq 3$, with the properties: $\varepsilon_i \varepsilon_j = 0$ for $i \neq j$ and $\varepsilon_1 + \dots + \varepsilon_n = 1$, then an isomorphism of $\mathfrak{C}(F)$ onto $\mathfrak{C}(G)$ is induced by a ring isomorphism of F onto G ((4), Part 2, p. 43, Theorem 4.2; (3), p. 192, Theorem 3.6). In the present note the question posed is solved for the case when the structure $\mathfrak{C}(F^n)$ is complete and continuous (a structure is called continuous if from $x_\alpha \uparrow x$ it follows that $ax_\alpha \uparrow ax$, and from $x_\alpha \downarrow x$ it follows that $a + x_\alpha \downarrow a + x$; see (1), p. 100).

By methods similar to Baer' s method ((2), pp. 62-70), one can obtain the following result:

Theorem 1. Let F and G be regular rings; $S \to S^*$ an isomorphism of $\mathfrak{C}(F^n)$ onto $\mathfrak{C}(G^m)$, $n \geq 3$; $[F(1, 0, \dots, 0)]^* = Ge'$; ε such an idempotent of the ring G that $N(e') = G(1 - \varepsilon)$; $H = \varepsilon G \varepsilon$. **Then in** G^m **there will be found elements** $e_1, \dots, e_n$ **such that** $N(e') = N(e_i)$**,** $[F(0, \dots, 0, 1, 0, \dots, 0)]^* = Ge_i$**,** $i = 1, 2, \dots, n$**, and also such a semilinear mapping** σ **of the** F**-module** F^n **onto the** H**-module** $\sum_1^n He_i$* that $(Fx)^* = Gx^\sigma$ for every $x \in F^n$.

If $m = n$ and $\mathfrak{C}(F^n)$ is complete and continuous, then the image $[F(1, 0, \dots, 0)]^*$ of the submodule $F(1, 0, \dots, 0)$ under an isomorphism of $\mathfrak{C}(F^n)$ onto $\mathfrak{C}(G^n)$ turns out to be perspective to $G(1, 0, \dots, 0)$. Hence one can infer that $[F(1, 0, \dots, 0)]^* = Ge'$, where $N(e') = 0$. Therefore Theorem 1 gives:

Theorem 2. If F and G are regular rings, $n \geq 3$, and the structure $\mathfrak{C}(F^n)$ is complete and continuous, then every isomorphism θ of the structure $\mathfrak{C}(F^n)$ onto the structure $\mathfrak{C}(G^n)$ is induced by a semilinear mapping σ of the module F^n onto the module G^n, i.e. $\theta(S) = \{\sigma(x);\, x \in S\}$ for every $S \in \mathfrak{C}(F^n)$.

---

[*] A module M over a ring F is called unitary if F contains the identity 1 and $1a = a$ for all $a \in M$. All modules discussed in the note are assumed to be left modules. An associative ring with identity is called regular if in it, for every a, the equation $axa = a$ is solvable.

[**] If $a \in G^m$, then by $N(a)$ is denoted the set of all elements $\lambda \in G$ such that $\lambda a = 0$. It can be shown that $N(a)$ is a principal left ideal of the ring G and therefore is generated by some idempotent.

[***] The definition of semilinear mapping for the case under consideration repeats verbatim Baer' s definition ((^{2}), p. 59).

If $\mathfrak{C}(F^n)$ is complete and continuous, and $\{S_1, \dots, S_m\}$ is an independent system of pairwise perspective elements of $\mathfrak{C}(F^n)^*$ and

$$\sum_1^m S_i = F^n,$$

then one can prove the existence in $\mathfrak{C}(F^n)$ of such an independent system $\{T_1, \dots, T_{\max\{m,n\}}\}$ of pairwise perspective elements that

$$F(1, 0, \dots, 0) \sim \sum_1^k T_i, \qquad S_1 \sim \sum_1^l T_i,$$

where $kn = lm$. This fact makes it possible to obtain the following result:

Theorem 3. Let F and G be regular rings, let $\mathfrak{C}(F^n)$ be complete and continuous, and let θ be an isomorphism of $\mathfrak{C}(F^n)$ onto $\mathfrak{C}(G^m)$, $3 \leq n < m$. Then there exist rings H and K such that H_k is isomorphic to F^{**}, K_l is isomorphic to G, and θ is induced by a semilinear mapping of H^{kn} onto K^{lm}.

Theorem 2 can be stated in another form:

Theorem 4. If F and G are regular rings, $n \geq 3$, and $\mathfrak{C}(F)$ is complete and continuous, then every isomorphism of F_n onto G_n is induced by some isomorphism of F onto G.

Indeed, if $F_n \leftrightarrow G_n$***, then, obviously, $\mathfrak{C}(F_n) \leftrightarrow \mathfrak{C}(G_n)$. But $\mathfrak{C}(F_n) \leftrightarrow \mathfrak{C}(F^n)$ and $\mathfrak{C}(G_n) \leftrightarrow \mathfrak{C}(G^n)$ ((^{3}), p. 186, Theorem 3.2), i.e. $\mathfrak{C}(F^n) \leftrightarrow \mathfrak{C}(G^n)$. Applying Theorem 2, we obtain that $F \leftrightarrow G$. It is not difficult to establish that the isomorphism obtained induces the given isomorphism of F_n onto G_n.

Theorem 3 can be reformulated in exactly the same way.

Moscow State University
named after M. V. Lomonosov

Received
1 XII 1959

REFERENCES

[1] G. Birkhoff, *Lattice Theory*, IL, 1952. [2] R. Baer, *Linear Algebra and Projective Geometry*, IL, 1955. [3] F. Maeda, *Kontinuierliche Geometrie*, Berlin–Göttingen–Heidelberg, 1958. [4] J. Neumann, *Lectures on Continuous Geometries*, Michigan, 1936–37.

---

[*] The elements $a_1, \dots, a_n$ of a lattice are said to be independent if

$$a_i \sum_{k \neq i} a_k = 0$$

for every i. Elements a and b are called perspective (notation: $a \sim b$) if they have a common complement (see ([1]), pp. 114 and 172).

[**] F_n denotes the ring of square matrices of order n with entries from the ring F.

[***] The symbol $A \leftrightarrow B$ denotes that A is isomorphic to B.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.