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Abstract

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V. A. Skripkin

A Point Explosion in an Ideal Incompressible Fluid in General Relativity

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In Taub' s paper ([1]) a general nonstationary solution of A. Einstein' s gravitational equations is given for the case of a plane-symmetric field produced by an ideal incompressible fluid. In particular, it is asserted in that paper that if the world line of some particle of the continuous medium belongs to the boundary between matter and vacuum, and the gravitational field outside the matter is static, then inside the region occupied by the fluid the field is also static. In the present paper a general solution of A. Einstein' s equations is given for an incompressible fluid in the presence of a centrally symmetric field. When the moving fluid borders on vacuum along the world lines of particles of two surfaces, the motion is interpreted as an explosion in an incompressible fluid. The transition to the Newtonian approximation is given. An analogous motion can also exist in the case of a plane-symmetric field.

1. Let us consider the radial motion of a gravitating ideal incompressible fluid, initially in equilibrium, caused by a sudden release of energy at the center of a sphere. In such a motion, a vacuum spherical cavity is formed in the central region of the sphere, so that the pressure $p = 0$ both at the outer and at the inner boundary of the moving fluid. We refer the motion to a Lagrangian orthogonal coordinate system, in which the square of the line element of 4-space $ds^2 = g_{ik}\,dx^i\,dx^k$ takes the form

$$ds^2 = c^2 e^{\nu(t,a)}\,dt^2 - e^{\lambda(t,a)}\,da^2 - e^{\mu(t,a)}(d\theta^2 + \sin^2\theta\,d\Phi^2), \qquad (1,1)$$

where $x^0 = t$ is the time coordinate, and $(x^1, x^2, x^3) = (a, \theta, \Phi)$ are the spatial coordinates of a point; c is the speed of light in vacuum. Then A. Einstein' s gravitational equations will have the form ([1])

$$\chi p = \frac{1}{2}e^{-\lambda}(\mu'^2/2 + \mu'\nu') - e^{-\nu}c^{-2}(\ddot{\mu} - \dot{\mu}\dot{\nu}/2 + 3\dot{\mu}^2/4) - e^{-\mu}; \qquad (1,2)$$

$$\chi p = \frac{1}{4}e^{-\lambda}(2\nu'' + \nu'^2 + 2\mu'' + \mu'^2 - \mu'\lambda' - \nu'\lambda' + \mu'\nu') +$$

$$+\frac{1}{4}e^{-\nu}c^{-2}(\dot{\lambda}\dot{\nu}+\dot{\mu}\dot{\nu}-\dot{\lambda}\dot{\mu}-2\ddot{\lambda}-\dot{\lambda}^2-2\ddot{\mu}-\dot{\mu}^2); \qquad (1,3)$$

$$-\chi\rho_0 c^2 = e^{-\lambda}(\mu''+3\mu'^2/4-\mu'\lambda'/2)-\frac{1}{2}e^{-\nu}c^{-2}(\dot{\lambda}\dot{\mu}+\dot{\mu}^2/2)-e^{-\mu}; \qquad (1,4)$$

$$0=\nu'\dot{\mu}+\dot{\lambda}\mu'-\dot{\mu}\mu'-2\dot{\mu}', \qquad (1,5)$$

where a prime denotes differentiation with respect to a, a dot with respect to t; $\chi = 8\pi G/c^4$, G is Newton' s constant; ρ_0 is the residual proper density of the incompressible–

fluid. From equations (1.2)–(1.5) follow the relations [(1)]

$$\dot{\lambda}+2\dot{\mu}=0, \qquad \nu'=-2p'/(p+\rho_0 c^2). \qquad (1.6)$$

2. Let $a = a_0 =$ const be the law of motion of the outer boundary of the fluid. Introduce the dimensionless variables

$$\tau=\sqrt{8\pi G\rho_0}\,t, \qquad \alpha = ae^{-\mu(0,a_0)/2}, \qquad \alpha_0 = a_0 e^{-\mu(0,a_0)/2}. \qquad (2.1)$$
$$R(\tau,\alpha)=e^{[\mu(t,a)-\mu(0,a_0)]/2}, \qquad f(\tau)=R(\tau,\alpha_0).$$

From the first equation (1.6) we find

$$e^{\lambda}=\varphi_1(\alpha)R^{-4}, \qquad (2.2)$$

where $\varphi_1(\alpha)$ is an arbitrary function. As a result of substituting (2.2) into (1.5) and integrating we obtain

$$e_{\nu}=R^4\dot{R}^2/\varphi_2(\tau), \qquad (2.3)$$

where $\varphi_2(\tau)$ is an arbitrary function. Without restricting generality, set

$$\varphi_1(\alpha)=1, \quad \varphi_2(\tau)=f^4\dot{f}^2. \qquad (2.4)$$

The equalities (2.4) finally fix the coordinate system to the gravitational field; moreover, the indicated form of $\varphi_2(\tau)$ corresponds to the choice of the proper time of the particle $\alpha=\alpha_0$. Equation (1.4), after eliminating λ and ν by means of (2.2)–(2.4), is transformed to the form

$$2R''+5{R'}^2R^{-1}+3\omega R^{-3}-R^{-5}+3\omega f^4\dot{f}^2R^{-9}=0, \qquad (2.5)$$

where $\omega = 4\beta_0(2\beta_0+1)(3\beta_0+1)^{-2}$, $\beta_0 = p_0/\rho_0 c^2$, $p_0 = p(0,0)$ at equilibrium [2]. From (2.5) we obtain

$$R'^2 = A(\tau)R^{-5} + \omega f^4 \dot{f}^2 R^{-8} + R^{-4} - \omega R^{-2}. \tag{2.6}$$

To determine the arbitrary function $A(\tau)$, we proceed as follows. From the second equation (1.6), taking into account that $p = 0$ for $R = f(\tau)$, we find

$$p\beta_0/p_0 = f^2\dot{f}/R^2\dot{R} - 1. \tag{2.7}$$

Substituting (2.7) and (2.2)–(2.4) into equation (1.2) and determining R'^2 from the equation obtained as a result of a single integration with respect to τ, we require the right-hand side of the expression for R'^2 to coincide with (2.6). This is satisfied when $A(\tau) = \omega(f^3 - 1)$. Integrating now equation (2.6), we shall have

$$\alpha - \alpha_0 = \int_R^f x^4\{x^4 + \omega[f^4\dot{f}^2 + (f^3-1)x^3 - x^6]\}^{-1/2}\,dx. \tag{2.8}$$

(2.8) determines R as a function of τ and α. If $\alpha = \alpha_1$ is the law of motion of the boundary of the inner cavity, then from (2.7) we obtain $p(\tau, \alpha_1) = 0$, $R^2\dot{R} = f^2\dot{f}$, whence

$$R(\tau, \alpha_1) = (f^3 - 1)^{1/3}. \tag{2.9}$$

Putting $\alpha = \alpha_1$ in (2.8) and taking (2.9) into account, we arrive at the differential equation for determining the function $f(\tau)$:

$$\alpha_1 - \alpha_0 = \Delta\alpha = \int_{(f^3-1)^{1/3}}^f x^4\{x^4 + \omega[f^4\dot{f}^2 + (f^3-1)x^3 - x^6]\}^{-1/2}\,dx. \tag{2.10}$$

For $\tau = 0$, $f = 1$, $\dot{f} = \dot{f}_0$, (2.10) gives the dependence of $\Delta\alpha$ on $\dot{f}_0$. If $\dot{f}_0 = 0$,

$$\Delta\alpha_0 = \int_0^1 x^2(1 - \omega x^2)^{-1/2}\,dx = (\arcsin\sqrt{\omega})/2\omega^{3/2} - (1-\omega)^{1/2}/2\omega.$$

As $\dot{f}_0 > 0$ increases, $\Delta\alpha$ decreases monotonically. The shortening of the interval $\Delta\alpha$ is an analogue of the Lorentz contraction of the length of a segment when the velocity of motion is increased.

Fig. 1

Fig. 1

Figure 1: Fig. 1

As $f_0 \to \infty$, $\Delta\alpha \to 0$. The velocity of motion of the fluid then tends to the speed of light. Indeed, the motion takes place in the plane $\theta = \text{const}$, $\Phi = \text{const}$, and since for an observer moving together with the particle $d\alpha = 0$, the square of the line element

$$ds^2 = g_{00}\, dt^2 = (c^2 R^4 \dot{R}^2 / f^4 \dot{f}^2)\, dt^2$$

tends to zero as $\dot{f} \to \infty$.*

In Fig. 1, 1, the graph is given of the dependence of $\Delta\alpha$ on $\dot{f}_0\sqrt{\omega}$ for $\omega = 0.01$, calculated from (2.10). Let us denote by f_m the maximum deviation from the initial position $f = 1$. Setting $\dot{f} = 0$ in (2.10), we obtain the dependence of f_m on $\Delta\alpha$ in the form

$$\Delta\alpha = \int_{(f_m^3-1)^{1/3}}^{f_m} x^3 \{x[\omega(f_m^3 - 1)] + x - \omega x^3\}^{-1/2}\, dx. \tag{2.11}$$

In Fig. 1, 2, the dependence of $\Delta\alpha$ on f_m is presented for $\omega = 0.01$. For $f_m = 10$, $\Delta\alpha = 0.333372$. For large f_m,

$$\Delta\alpha \simeq \frac{1}{3} + \frac{\omega}{12 f_m} + \frac{\omega^2}{32 f_m^2} + \cdots.$$

Thus, the value $\Delta\alpha = 1/3$ corresponds to such a value of $\dot{f}_0$ (see Fig. 1) that the fluid will fly apart to infinity. For $\Delta\alpha_0 > \Delta\alpha > 1/3$ the fluid will execute motion of an oscillatory type, consisting of expansion, stopping, and collapse.

3. If in the motion under consideration the values of the particle velocities are small, and the parameter ω, proportional to the quantity $\beta_0 = p_0/\rho_0 c^2$, is very small, then, neglecting terms with ω in (2.10), we find for the law of motion of any particle the expression $R = (f^3 - \alpha)^{1/3}$; $\alpha = 0$ corresponds to $R = f(\tau)$—the outer boundary; $\alpha = 1$ corresponds to $R = (f^3 - 1)^{1/3}$ —the inner boundary. Expanding in series in powers of β_0 the function $p/p_0 =$

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* $\infty > g_{00} > 0, \quad \dot{f} = \infty, \quad$ when $dt = 0$.

$= p_1/p_0 + p_2\beta_0/p_0 + ...$ and the integrand in (2.8), we obtain from (2.7), for the pressure in the Newtonian approximation, the formula

$$p_1/p_0 = -2f^4\dot{f}^2R^{-4} - 2(f^3-1)/R + 2R^2 - 4(f^2\dot{f})^{\cdot}(R-f)/Rf$$

Fig. 2

Figure 2: Fig. 2

$$- 3(R^2 - f^2) + 2(f\dot{f}^2 - 1)/f.$$

In an analogous way, in the Newtonian approximation the differential equation (2.10) for $f(\tau)$ takes the form

$$\dot{f}^2 = (f^3 - 1)^{1/3}\{f^2/2 - 3f^5/10 + 3(f^3 - 1)^{5/3}/10 \\ - 2(\Delta\alpha - 1/3)/\omega\}/f^3[f - (f^3 - 1)^{1/3}]. \quad (3.1)$$

The constant $2(\Delta\alpha - 1/3)/\omega$ is related to the explosion energy E_0 by the dependence

$$2(\Delta\alpha - 1/3)/\omega = 1/5 - E_0/16\pi^2 G\rho_0^2 r_0^5,$$

where r_0 is the radius of the sphere in equilibrium.

Fig. 2

Let us note that in the Newtonian approximation, at $t = 0$ and $f = 1$, in contrast to general relativity, $\dot{f}_0 = 0$. Denoting by $\varepsilon_0 = E_0/16\pi G\rho_0^2 r_0^5$, from equation (3.1) we find

$$\tau = \int_1^f \frac{x^{3/2}\left[x - (x^3 - 1)^{1/3}\right]^{1/2} dx}{(x^3 - 1)^{1/6}\left[\varepsilon_0 - 1/5 + 3(x^3 - 1)^{5/3}/10 - 3x^5/10 + x^2/2\right]^{1/2}}. \quad (3.2)$$

Figure 2 shows the graph of the function $\Phi(x) = -1/5 + 3(x^3 - 1)^{5/3}/10 - 3x^5/10 + x^2/2$. As $x \to \infty$, $\Phi(x) \to -1/5$. The quantity ε_0 determines the upper limit of integration f in (3.2). When $\varepsilon_0 \geq 1/5$, the fluid expands to infinity. Figure 2 also presents the graph of the law of motion of the particles of the outer boundary of the fluid when $\max f = f_m = 3$. The corresponding value of ε_0 for $f_m = 3$ is marked on the graph.

Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.