MATHEMATICS

A. A. DEZIN

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. A. DEZIN

BOUNDARY-VALUE PROBLEMS FOR INVARIANT ELLIPTIC SYSTEMS

(Presented by Academician I. N. Vekua on 27 II 1960)

In the paper (^{1}) a class of systems was described that admit consideration on an arbitrary Riemannian manifold. We are now interested in describing correct boundary-value problems for these systems, regarded as first-order systems with constant coefficients in a finite domain V of Euclidean space. In doing so, however, the invariant character of the systems is used essentially.

Let the domain V be referred to coordinates $x^1, \dots, x^n$. We shall consider in V systems of functions $\overset{p}{\omega}$ $(p = 0, 1, \dots, n)$, each of which contains C_n^p components (C_n^p is a binomial coefficient). We shall call $\overset{p}{\omega}$ a covariant. The components of each covariant (for $p \geqslant 1$) are numbered by systems of indices $i_1 < i_2 < \cdots, i_p$, $1 \leqslant i_k \leqslant n$, and are ordered lexicographically, i.e. $\omega_{i_1 \dots i_p}$ precedes $\omega_{j_1 \dots j_p}$ if, for the first of the indices j_l for which $j_l \neq i_l$, we have $i_l < j_l$. Considering the components as functions of class C^1, we define a differential operator d assigning to $\overset{p}{\omega}$ the covariant $d\overset{p}{\omega} = \overset{p+1}{\alpha}$, whose components are formed according to the rule

$$(d\overset{p}{\omega})_{j_1 \dots j_{p+1}} = \sum \varepsilon D_{i_l} \omega_{i_1 \dots i_p}; \qquad D_{i_l} \equiv \frac{\partial}{\partial x^{i_l}}, \tag{1}$$

where all permutations $i_l i_1 \dots i_p$ (in number $p+1$) over which the summation is carried out are obtained from $j_1 \dots j_{p+1}$ by moving one of the indices to the first place. The quantity ε is equal to $+1$ or -1 according as the number of transpositions returning the index to its original place is even or odd. If to the covariant we assign the differential form

$$\overset{p}{\omega} = \sum \omega_{i_1 \dots i_p} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \tag{2}$$

then the connection of the operator introduced with exterior differentiation is clear.

Define in V the scalar product of covariants by putting

$$(\overset{p}{\omega}, \overset{p}{\chi}) = \int\limits_V [\overset{p}{\omega}, \overset{p}{\chi}]\, dV, \tag{3}$$

where $[\overset{p}{\omega}, \overset{p}{\chi}]$ is the sum of the pairwise products of components with identical systems of indices. If the components vanish on the boundary of V, then the relation

$$(d\overset{p-1}{\omega}, \overset{p}{\omega}) = (\overset{p-1}{\omega}, \delta\overset{p}{\omega}), \tag{4}$$

obtained by transferring the differentiations from the components of $\overset{p-1}{\omega}$ to the components of $\overset{p}{\omega}$ by integration by parts, uniquely determines the operator δ.

For completeness, we give the notation for the systems of interest to us in the case of even n (^{1}):

$$\begin{aligned} d\omega^0 + \delta\omega^2 &= \alpha^1, \\ d\omega^2 + \delta\omega^4 &= \alpha^3, \\ &\cdots \qquad (K) \\ d\omega^{n-2} + \delta\omega^n &= \alpha^{n-1}; \end{aligned} \qquad \begin{aligned} \delta\omega^1 &= \alpha^0 \\ d\omega^1 + \delta\omega^3 &= \alpha^2, \\ &\cdots \qquad (K^*) \\ d\omega^{n-3} + \delta\omega^{n-1} &= \alpha^{n-2}, \\ d\omega^{n-1} &= \alpha^n. \end{aligned}$$

If the aggregate of covariants of even degree is denoted by $\ddot{\omega}$, and that of odd degree by $\dot{\omega}$, and the above systems are written briefly as

$$L\ddot{\omega} = \dot{\alpha}; \qquad (K) \qquad L^*\dot{\omega} = \ddot{\alpha}, \qquad (K^*)$$

then, understanding by $(\dot{\omega}, \dot{\chi})$ the sum of the scalar products (3) for odd p, we obtain, according to (4),

$$(L\ddot{\omega}, \dot{\omega}) = (\ddot{\omega}, L^*\dot{\omega}),$$

i.e. the systems (K), (K^*) are formally adjoint.

Let now V be homeomorphic to an n-dimensional ball and have a smooth boundary. Consider a second copy V' of the basic domain, assigning V' the coordinates $y^1, \dots, y^n$. Points of V, V' whose coordinates, when V, V' are referred to one and the same coordinate system, coincide, we shall call corresponding. We introduce the coordinates (y) in V' in such a way that for corresponding points

$$y^1 = -x^1, \qquad y^i = x^i, \qquad i = 2, \dots, n.$$

Regarding V, V' as distinct domains and identifying corresponding boundary points, we obtain a closed n-dimensional manifold M, homeomorphic to the n-dimensional sphere. The submanifold consisting of the identified points of the boundaries of V, V' will be called the edge. Suppose that a neighborhood of the boundary in V is covered by a finite number of domains γ_σ such that

$$x^i = f^i_\sigma(\xi), \qquad \xi = (\xi^1, \dots, \xi^n), \qquad i = 1, \dots, n, \tag{5}$$

is a smooth mapping (one-to-one, with Jacobian bounded away from zero) of the half-ball

$$\sum (\xi^i)^2 < 1, \qquad \xi^1 \leq 0, \tag{6}$$

onto γ_σ, and in the coordinates (ξ) the equation of the portion of the boundary of V has the form $\xi^1 = 0$. If γ'_σ is the corresponding neighborhood in V', then the system of functions

$$y^1 = -f^1_\sigma(-\xi^1, \xi^2, \dots, \xi^n), \qquad y^i = f^i_\sigma(-\xi^1, \xi^2, \dots, \xi^n), \qquad i = 2, \dots, n,$$

gives a mapping onto γ'_σ of the half-sphere complementary to (6). The original coordinates $(x), (y)$, together with the coordinates (ξ_σ), give a complete covering of M by coordinate neighborhoods. With this covering, in particular, a decomposition of unity (^{2}) may be associated. The Euclidean metric in V, V' induces a metric in M; moreover, if one requires that the metric tensor in the coordinates (ξ) satisfy the relations $g_{1i} = 0$, $i = 2, \dots, n$ (from the geometric point of view, that the coordinate ξ^1 be the distance along the normal to the boundary of V), then the metric will be Lipschitz-continuous—discontinuous in a neighborhood of the edge (the first derivatives, generally speaking, undergo a jump; cf. (3)). Thus M becomes a Riemannian manifold. Carrying out the consideration on M, we shall assume that a covariant $\overset{p}{\omega}$ is the collection of essential components of a skew-symmetric tensor of rank p, or, equivalently, the collection of coefficients of a differential form. Then on $\overset{p}{\omega}$ the operators d and δ and the scalar product $(\overset{p}{\omega}, \overset{p}{\chi})$ are defined, having meaning in any neighborhood of M (cf. (2)) and coinciding in the coordinates (x) with the operations introduced by relations (1), (4), (3). Defining on M the norms

$$|\overset{p}{\omega}, H|^2 = (\overset{p}{\omega}, \overset{p}{\omega}), \qquad |\overset{p}{\omega}, W|^2 = (d\overset{p}{\omega}, d\overset{p}{\omega}) + (\delta\overset{p}{\omega}, \delta\overset{p}{\omega})$$

and completing, with respect to these norms, the set of smooth covariants, we obtain Hilbert spaces H and W. We shall call a collection $\ddot{\omega} \in H$ (the meaning of the inclusion is obvious) a generalized solution of the system (K) on M, if there exists $\dot{\alpha} \in H$ such that for every $\dot{\omega} \in C^1$

$$(\dot{\omega}, L^{*}\dot{\omega}) = (\dot{\alpha}, \dot{\omega}).$$

For what follows it is convenient to exclude constants from consideration, subjecting covariants of degrees 1 and n to the requirements

$$\int_M \overset{0}{\omega} \wedge *1 \equiv \int_M \overset{0}{\omega}\, dV = 0, \qquad \int_M \overset{n}{\omega} \equiv \int_M \overset{n}{\omega}\, dV = 0.$$

Lemma 1. For generalized solutions of the systems (K), (K*) on M, the equalities

$$|\ddot{\omega}, W| = |L\ddot{\omega}, H|, \qquad |\dot{\omega}, W| = |L^{*}\dot{\omega}, H|$$

hold

(the extension of the definition of the norm to the collections $\dot{\omega}$, $\ddot{\omega}$ is obvious).

In proving the lemma, the main role is played by the equivalence of the so-called "weak" and "strong" extensions of the operators d and δ (cf. (3)) and by the presence in H of orthogonal decompositions—variants of Kodaira' s theorem. From the lemma, in the usual way, follows the theorem:

Theorem 1. A generalized solution of the systems (K), (K*) for any right-hand side from H exists, is unique, and belongs to W.

To pass from considerations on M to the domain V, let us introduce special subspaces of covariants. Consider M as the sum $V + V'$, assuming that V is referred to coordinates (x), and V' to (y). We shall denote systems of indices by one letter i. We shall call the component ω_i **normal** if the system of indices contains 1 $(1 \in i)$, and **tangential** in the contrary case $(1 \notin i)$. We require that, for the components at corresponding points of V, V', the systems of equalities hold: either

$$\omega_i(y) = \omega_i(x), \quad 1 \notin i; \qquad \omega_i(y) = -\omega_i(x), \quad 1 \in i; \tag{7}$$

or

$$\omega_i(y) = -\omega_i(x), \quad 1 \notin i; \qquad \omega_i(y) = \omega_i(x), \quad 1 \in i. \tag{8}$$

We shall call covariants satisfying conditions (7) **even**, and those satisfying conditions (8) **odd**. Let now $\overset{p}{\omega}$ be a system of smooth functions given in V. Assuming that $\overset{p}{\omega}$ defines an even covariant on M, and clarifying the conditions ensuring continuity of ···

of ray covariance in a neighborhood of an edge, we arrive at the relations

$$D_{i'} f^i(\xi)\,\omega_i(x(\xi)) = 0, \quad 1 \in i'; \qquad D_{i'} = D_{i'_1} \dots D_{i'_p}; \qquad D_i \equiv \frac{\partial}{\partial \xi^i}, \tag{9}$$

which must be satisfied by the components of the system extended according to (7). Fixing the choice of the functions (5), we obtain a definite system of boundary conditions, which can be represented in a notation containing only functions of the original coordinates (x) in V. The norms H and W are defined in the natural way also in V. Completing, in the norm W, the set of smooth covariants given in V and satisfying the conditions (9), we obtain the Hilbert space $W_{\mathrm{q}}(V)$. The boundary conditions (on the average) for elements of this space retain their meaning by virtue of embedding theorems [4]. By a generalized solution of the system (K) under the boundary conditions (9) we shall mean a collection of covariants $\ddot{\omega}$ from $W_{\mathrm{q}}(V)$ satisfying the equations (K) in the sense of equality in H. Using the solutions constructed on M, one can establish the theorem:

Theorem 2. *A generalized solution of the system (K) under the boundary conditions (9) exists and is unique for any right-hand side from H.*

The corresponding assertions for the system (K^*) and for the conditions connected with odd covariants are formulated analogously. We note that the number of conditions (9) is always equal to one half the number of unknown functions entering the system (K).

The passage to domains V whose boundary is only piecewise smooth requires a rather substantial modification of the constructions given above (reflections are used).

In an infinite domain, for $n = 3$, an explicit solution of a number of special problems for systems (K) was constructed by A. V. Bitsadze [5,6]. The classical case $(n = 2)$ has, as is well known, been studied exhaustively.

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
24 II 1960

CITED LITERATURE

[1] A. A. Dezin, DAN, **130**, No. 6 (1960).
[2] G. de Rham, *Differentiable Manifolds*, IL, 1956.
[3] K. O. Friedrichs, Comm. Pure and Appl. Math., **8**, 551 (1955).
[4] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
[5] A. V. Bitsadze, DAN, **93**, No. 3 (1953).
[6] A. V. Bitsadze, Soobshch. AN GruzSSR, **16**, No. 3 (1955).

Note: Figure translations are in progress. See original paper for figures.