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Abstract

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FUNCTIONS OF MANY VARIABLES WITH RAPIDLY CONVERGENT HAAR SERIES

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In the article [1] some results from the theory of quadrature formulas are given, obtained by means of a new (for this field) method—Haar series. In the present paper classes of functions S_p are introduced, playing an important role in the further development of the method. Also considered are classes of functions H_α, which may be regarded as multidimensional analogues of Lipschitz classes, and their relation to the classes S_p.

1. The Haar system. The complete orthonormal system of Haar functions $\{\chi_k(x)\}$ on $[0,1]$ is conveniently constructed by groups [2]. The group numbered m contains 2^{m-1} functions $\{\chi_{mj}(x)\}$, $j = 1, 2, \dots, 2^{m-1}$; $m = 1, 2, \dots$. The relation between the double numbering (m, j) and the ordinary one (k) is given by the formula

$$k = 2^{m-1} + j,$$

where the first function $\chi_1(x) \equiv 1$ remains outside the groups.

Definition.

$$\chi_{mj}(x) = \begin{cases} 2^{(m-1)/2}, & \text{if } x \in l_{mj}^+, \\ -2^{(m-1)/2}, & \text{if } x \in l_{mj}^-, \\ 0, & \text{if } x \notin l_{mj}. \end{cases}$$

Here the intervals $l_{mj} \equiv l_k = [(j-1) \cdot 2^{-(m-1)},\ j \cdot 2^{-(m-1)}]$, and l_{mj}^+ and l_{mj}^- are the left and right halves of l_{mj}.

We shall agree to regard all intervals occurring in the exposition as closed on the left and open on the right, unless the right endpoint is equal to 1. If the right endpoint is equal to 1, then we shall regard the interval as closed also on the right. Thus,

$$\sum_j l_{mj} = [0, 1].$$

With this definition of the functions $\chi_{mj}(x)$, the Haar series of any continuous function converges to it uniformly on $[0,1]$ (one may use the proof given in [2]).

In order not to single out each time the function $\chi_1(x)$, we shall agree to regard it as the first (and only) function of the zero group, $\chi_1(x) \equiv \chi_{01}(x)$. Introduce the auxiliary symbol $\widetilde{m}$:

$$\widetilde{m} = 1 \quad \text{for } m = 0; \qquad \widetilde{m} = m \quad \text{for } m \geq 1.$$

With the help of $\widetilde{m}$ one can write some relations needed by us in such a way that they also include the case $m = 0$:

$$\chi_{mj}(x) = 2^{(\widetilde{m}-1)/2} \operatorname{sgn} \chi_{mj}(x); \tag{1}$$

$$\sum_j |\chi_{mj}(x)| = 2^{(m-1)q/2}. \tag{2}$$

The number of functions in group m is equal to $2^{\widetilde{m}-1}$.

2. The class of functions S_p

All possible products

$$\chi_{k_1}(x_1)\chi_{k_2}(x_2) \cdots \chi_{k_d}(x^d)$$

form a complete orthonormal system on the unit d-dimensional cube $K : 0 \leq x_s \leq 1,\ s = 1, 2, \ldots, d$. As in the one-dimensional case, the expansion in the Haar system of any continuous function $f(P) = f(x_1, \ldots, x_d)$ converges uniformly.

Let p be an arbitrary number, $1 \leq p < \infty$. For functions $f(x_1, \ldots, x_d)$ representable in K by Haar series

$$f(x_1, \ldots, x_d) = \sum c_{k_1 \ldots k_d} \chi_{k_1}(x_1) \ldots \chi_{k_d}(x_d), \tag{3}$$

we define the norm

$$\|f\|_p = \sum_{m_1, \ldots, m_d}' 2^{(\widetilde{m}_1-1)/2 + \ldots + (\widetilde{m}_d-1)/2} \left\{ \sum_{j_1, \ldots, j_d} |c_{k_1 \ldots k_d}|^p \right\}^{1/p}. \tag{4}$$

The prime, as usual, indicates that the case $m_1 = \cdots = m_d = 0$ is excluded from the summation. Clearly, functions differing by a constant term have identical norms.

Definition. S_p is the set of functions $f(x_1, \dots, x_d)$, representable in the form of the series (3), for which $\|f\|_p < \infty$.

If functions differing only by a constant term are identified, then S_p becomes a complete linear normed space.

Theorem. *Functions of the class S_p are continuous at all points of K, except possibly at dyadic-rational points* *.

For the proof of the theorem it suffices to verify that, if $f(x_1, \dots, x_d) \in S_p$, then the series (3) converges uniformly. Let $1/p + 1/q = 1$. By Hölder' s inequality**

$$\sum_j |c_i \chi_{k_1}(x_1) \dots \chi_{k_d}(x_d)| \le \left\{ \sum_j |c|^p \right\}^{1/p} \left\{ \sum_j |\chi_{k_1}(x_1) \dots \chi_{k_d}(x_d)|^q \right\}^{1/q}.$$

Computing the second bracket by means of formula (2), it is easy to show that the series (3) is majorized by the numerical series (4).

The class S_p contains all possible finite linear combinations of Haar functions. These are, generally speaking, discontinuous piecewise-constant functions. Below it will be proved that S_p contains a very broad class of continuous functions.

3. The class of functions H_α

Consider again functions $f(P) = f(x_1, \dots, x^d)$ defined on the unit cube K. Denote the increment of the argument x_s by ξ_s and introduce the usual difference operator

$$\Delta_{\xi_s} f(P) = f(x_1, \dots, x_s + \xi_s, \dots, x_d) - f(x_1, \dots, x_d).$$

Definition. H_α, where $0 < \alpha \le 1$, is the set of functions satisfying in K the following conditions:

$$|\Delta_{\xi_s} f(P)| \le \frac{\alpha+1}{2} L |\xi_s|^\alpha;$$

$$|\Delta_{\xi_s} \Delta_{\xi_t} f(P)| \le \left(\frac{\alpha+1}{2}\right)^2 L |\xi_s \xi_t|^\alpha, \qquad s \ne t; \tag{5}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$|\Delta_{\xi_1} \cdots \Delta_{\xi_d} f(P)| \le \left(\frac{\alpha+1}{2}\right)^d L |\xi_1 \cdots \xi_d|^\alpha.$$

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* We shall call a point $(x_1, \dots, x_d)$ dyadic-rational if any of its coordinates x_s is dyadic-rational.

** Here, for brevity, $k = (k_1, \dots, k_d)$, $j = (j_1, \dots, j_d)$.

An important special case is the class H_1: it consists of functions with bounded first mixed derivatives, where

$$L, \quad \sup_{K} \max_{(s \neq t \dots)} \left\{ \left| \frac{\partial f}{\partial x_s} \right| ; \left| \frac{\partial^2 f}{\partial x_s \partial x_t} \right| ; \dots ; \left| \frac{\partial^d f}{\partial x_1 \dots \partial x_d} \right| \right\}.$$

In the one-dimensional case (for $d = 1$), H_α is the class $\mathrm{Lip}\,\alpha$.

The following simple lemma makes it possible to estimate increments of functions of the class H_α:

Lemma. If $P = (x_1, \dots, x_d)$, $Q = (\xi_1, \dots, \xi_d)$, then

$$f(P+Q) = f(P) + \sum_{s} \Delta_{\xi_s} f(P) + \sum_{s<t} \Delta_{\xi_s} \Delta_{\xi_t} f(P) +$$

$$+ \sum_{s<t<u} \Delta_{\xi_s} \Delta_{\xi_t} \Delta_{\xi_u} f(P) + \dots + \Delta_{\xi_1} \cdots \Delta_{\xi_d} f(P). \tag{6}$$

4. Embedding of H_α in S_p.

Theorem. If $\alpha p > 1$, then $H_\alpha \subset S_p$. Moreover,

$$\|f\|_p \le L \left[\left(1 + \frac{0.5}{2^\alpha - 2^{1/p}} \right)^d - 1 \right]. \tag{7}$$

To prove this theorem, it is necessary to estimate the expansion coefficients of an arbitrary function $f(P) \in H_\alpha$ with respect to the Haar system:

$$c_{k_1 \dots k_d} = \int_0^1 \cdots \int_0^1 f(P) \chi_{k_1}(x_1) \cdots \chi_{k_d}(x_d)\, dP. \tag{8}$$

Let us outline the proof.

1) Suppose first that all $k_s \neq 1$. Since $\chi_{k_s} \equiv 0$ outside l_{k_s}, we may assume that in the integral (8) the domain of integration is the parallelepiped $\Pi_{k_1 \dots k_d} = l_{k_1} \times \dots \times l_{k_d}$. Move the origin of coordinates to the center P' of this parallelepiped, and denote the new coordinates by ξ_s: $Q = P - P'$. If $x_s \in l_{k_s}$, then $\operatorname{sgn} \chi_{k_s}(x_s) = -\operatorname{sgn} \xi_s$. Expanding $f(P'+Q)$ by formula (6), we see that all terms except the last give zeros upon integration. Therefore

$$c_{k_1\dots k_d} = (-1)^d 2^{(\widetilde{m}_1-1)/2+\dots+(\widetilde{m}_d-1)/2} \int \dots \int \Delta_{\xi_1} \cdots \Delta_{\xi_d} f(P') \operatorname{sgn}(\xi_1 \dots \xi_d)\, dQ, \tag{9}$$

where integration with respect to ξ_s is carried out from -2^{-m_s} to 2^{-m_s}. With the aid of the last of conditions (5), we obtain the estimate

$$|c_{k_1\dots k_d}| \le L \prod_{s=1}^{d} 2^{(\widetilde{m}_s-1)/2-m_s(\alpha+1)}. \tag{10}$$

2) Now let one of the k_s, say k_1, be equal to 1. The reasoning remains the same, but the expansion (6) must be applied only in the coordinates $\xi_2, \dots, \xi_d$. Instead of formula (9) we obtain

$$c_{1k_2\dots k_d} = (-1)^{d-1} 2^{(\widetilde{m}_2-1)/2+\dots+(\widetilde{m}_d-1)/2} \times$$

$$\times \int \dots \int \Delta_{\xi_2} \cdots \Delta_{\xi_d} f(x_1' + \xi_1, x_2', \dots, x_d') \operatorname{sgn}(\xi_2 \dots \xi_d)\, dQ.$$

Using the corresponding condition (5), we again arrive at estimate (10), since in the case under consideration $m_1 = 0$, $\widetilde{m}_1 = 1$.

3) Thus, estimate (10) is valid for all coefficients $c_{k_1\dots k_d}$. Substituting it into (4), we obtain a series (a product of geometric progressions with ratio $2^{-(\alpha-1/p)}$), whose sum appears in (7).

5. **Example.** Let $f(x) = Lx$. The expansion of this function is

$$f(x) = L\left\{0.5 - \sum_{m=1}^{\infty} \sum_{j=1}^{2^{m-1}} 2^{-(3m+1)/2} \chi_{mj}(x)\right\}.$$

The norm is easily computed:

$$\|f\| = L \frac{0.5}{2 - 2^{1/p}}.$$

This example shows that equality is possible in formula (7). From the same example it is clear that, in the conditions of the theorem, the requirement $\alpha p > 1$ cannot be replaced by $\alpha p \geqslant 1$.

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REFERENCES

1. I. M. Sobol, *DAN*, **114**, No. 4, 706 (1957).
2. S. Kaczmarz, H. Steinhaus, *Theory of Orthogonal Series*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.

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