ESTIMATING THE INFLUENCE OF THE SURFACE LAYER ON THE DEPOSITION OF A HEAVY ADMIXTURE FROM OVERLYING LAYERS OF THE ATMOSPHERE WITH WIND VARYING WITH HEIGHT

1960

SovietRxiv

Abstract

Full Text

GEOPHYSICS

O. S. BERLYAND and A. Ya. PRESSMAN

ESTIMATING THE INFLUENCE OF THE SURFACE LAYER ON THE DEPOSITION OF A HEAVY ADMIXTURE FROM OVERLYING LAYERS OF THE ATMOSPHERE WITH WIND VARYING WITH HEIGHT

(Presented by Academician A. A. Dorodnitsyn on 2 VI 1960)

In the work ([1]), the question of deposition and dispersion of a heavy admixture from an elevated source in a field of wind varying with height was investigated, where the wind velocity was specified in the form of a polynomial of arbitrary degree in the height z. The dispersion coefficients of the admixture were assumed constant.

In the present article an attempt is made to estimate the influence of the surface layer, in which vertical dispersion is limited by the presence of the underlying surface and is therefore less intense than in the free atmosphere, on the distribution of the admixture deposited on the ground, taking into account the variation of wind speed with height. For this purpose, the equation of turbulent diffusion

$$\frac{\partial q}{\partial t} + u(z)\frac{\partial q}{\partial x} - w\frac{\partial q}{\partial z} = k_x\frac{\partial^2 q}{\partial x^2} + k_y\frac{\partial^2 q}{\partial y^2} + k_z\frac{\partial^2 q}{\partial z^2} \tag{1}$$

(where $u(z)$ is the wind velocity variable with height, specified in the form

$$u(z) = \sum_{m=0}^{k} b_m z^m,$$

$w > 0$ is the constant falling velocity of the admixture particles) will be solved for an atmosphere divided by height into two layers: $z > H,\ k_z = k_1$, and $0 < z \leq H,\ k_z = k_2 < k_1$.

Consider an instantaneous point source placed at height $h > H$. The boundary conditions are prescribed for the different layers as follows. In the upper layer $q_1 \to 0$ as $\sqrt{x^2 + y^2 + z^2} \to \infty$; at the boundary of the layers, for $z = H$, $q_1 = q_2$ and

$$k_1 \frac{\partial q_1}{\partial z} = k_2 \frac{\partial q_2}{\partial z}$$

(whence it follows that

$$[wq_1 + k_1 \partial q_1/\partial z]_{z=H} = [wq_2 + k_2 \partial q_2/\partial z]_{z=H}$$

–the equality of fluxes at the boundary of the layers), and finally, for $z = 0$, $q_2 = 0$.

The initial conditions are given in the form: at $t = 0$,

$$q_1 = Q\delta(x)\delta(y)\delta(z-h), \qquad q_2 = 0$$

(here Q is the total amount of admixture in the source; δ is the Dirac delta function).

After the change of variables $\xi = x/\sqrt{k_x}$, $\eta = y/\sqrt{k_y}$ and the application of the Laplace transform (one-sided in t with parameter p_t, and two-sided in ξ and η with parameters p_ξ and p_η), equation (1) takes the form:

$$k_1 \frac{d^2\bar{q}_1}{dz^2} + w\frac{d\bar{q}_1}{dz} - p\bar{q}_1 = p_\xi u(z)\bar{q}_1 - \frac{Q}{\sqrt{k_{xk}y}}\delta(z-h) \quad \text{for } z \geq h; \tag{2}$$

$$k_1 \frac{d^2\bar{q}_1}{dz^2} + w\frac{d\bar{q}_1}{dz} - p\bar{q}_1 = p_\xi u(z)\bar{q}_1 \quad \text{for } H \leq z \leq h; \tag{3}$$

$$k_2 \frac{d^2\bar{q}_2}{dz^2} + w\frac{d\bar{q}_2}{dz} - p\bar{q}_2 = p_\xi u(z)\bar{q}_1 \quad \text{for } 0 \leq z \leq H. \tag{4}$$

Here

$$p = p_t - p_\xi^2 - p_\eta^2.$$

Solving system (2)–(4) by the method set forth in ([1]), we obtain

$$\bar{q}_1 = e^{-wz/2k_1}\left[C_1 Y_{11}(z) + C_2 Y_{12}(z)\right], \qquad q_2 = C^* e^{-wz/2k_2}\left[Y_{21}(z) - \frac{Y_{21}(0)}{Y_{22}(0)} Y_{22}(z)\right], \tag{5}$$

where

$$Y_{11}(z) = e^{\xi_1} F_{11}(\xi_1); \qquad Y_{12}(z) = e^{-\xi_1} F_{12}(\xi_1);$$

$$F_{11}(\xi_1) = \sum_{n=0}^{\infty} 2^{nk} p_\xi^n p_k^n k_x^{-n/2} k_1^{n(k+1)} \left[w^2 + 4pk_1\right]^{-n(k+2)/2} \chi_{n1}(\xi_1),$$

$$F_{12}(\xi_1) = \sum_{n=0}^{\infty} 2^{nk} p_\xi^n p_k^n k_x^{-n/2} k_1^{n(k+1)} \left[w^2 + 4pk_1\right]^{-n(k+2)/2} \chi_{n2}(\xi_1),$$

$$\xi_1 = \frac{z}{2k_1}\sqrt{w^2 + 4pk_1};$$

$\chi_n(\xi)$ are polynomials of degree $n(k+1)$ in ξ, obtained in ([1]) with the aid of a special operator. $Y_{21}(z)$ and $Y_{22}(z)$ are obtained from $Y_{11}(z)$ and $Y_{12}(z)$ by replacing k_1 with k_2.

The constants C_1, C_2, and C^*, determined from the boundary conditions, have the form

$$C_1 = -\frac{2QY_{12}(h)}{\sqrt{k_{xk}y}\sqrt{w^2 + 4pk_1}\,A(0)}\,e^{wh/2k_1},$$

$$C_2 = C_1 \frac{\sqrt{w^2 + 4pk_1}\,Y_{11}'(H)B(H) - \sqrt{w^2 + 4pk_2}\,Y_{11}(H)A_2(H)}{\sqrt{w^2 + 4pk_2}\,Y_{12}(H)A_2(H) - \sqrt{w^2 + 4pk_1}\,Y_{12}'(H)B(H)}, \qquad (6)$$

$$C^* = C_1 e^{Hw/2k_2} \frac{A_1(0)Y_{22}(0)\sqrt{w^2 + 4pk_1}}{\sqrt{w^2 + 4pk_2}\,Y_{12}(H)A_2(H) - \sqrt{w^2 + 4pk_1}\,Y_{12}'(H)B(H)},$$

where

$$A_1(H) = \begin{vmatrix} Y_{11}'(H) & Y_{12}'(H) \\ Y_{11}(0) & Y_{12}(0) \end{vmatrix}; \qquad A_2(H) = \begin{vmatrix} Y_{21}'(H) & Y_{22}'(H) \\ Y_{21}'(0) & Y_{22}'(0) \end{vmatrix};$$

$$B(H) = \begin{vmatrix} Y_{21}(H) & Y_{22}(H) \\ Y_{21}(0) & Y_{22}(0) \end{vmatrix};$$

the prime denotes differentiation with respect to the entire argument ξ.

The expression for the flux of the settling impurity at the surface $z = 0$, written in the form

$$\Pi_0 = k_2 \frac{d\bar{q}_2}{dz} = \frac{1}{2} C_1 e^{\frac{wH}{2}\left(\frac{1}{k_2} - \frac{1}{k_1}\right)} \frac{A_1(0)A_2(0)\sqrt{w^2 + 4pk_1}\sqrt{w^2 + 4pk_2}}{\sqrt{w^2 + 4pk_2}\,Y_{12}(H)A_2(H) - \sqrt{w^2 + 4pk_1}\,Y_{12}'(H)B(H)}, \qquad (7)$$

will be considered by us in the following limiting cases: 1) $0 < k_2 \leq k_1$; 2) $k_2 \simeq 0$, with the first case evidently equivalent to investigating (7) for small values of H, since for $k_2 = k_1$ the near-surface layer (in our sense) disappears.

Writing out two terms of the expansion of Π_0 in a Taylor series in small H, we obtain the principal term, not containing H, in which $k_2 = k_1$, and the second term, making it possible further to estimate the error from neglecting the decrease, in the near-surface layer, of the scattering coefficient k_2 when calculating the surface concentration of the heavy impurity that has settled on the ground:

$$\Pi_0 \simeq \Pi_0\big|_{H=0} + H\frac{\partial \Pi_0}{\partial H}\bigg|_{H=0} =$$

$$= \frac{2Q}{\sqrt{k_{xk}y}}\, e^{wh/2k_1}\, \frac{1}{A(0)}\frac{Y_{12}(h)}{Y_{12}(0)}\left[1 + \left(\frac{k_1}{k_2} - 1\right)\frac{H}{\sqrt{k_1}}\sqrt{\frac{w^2}{4k_1} + p}\,\frac{Y_{12}'(0)}{Y_{12}(0)}\right]. \tag{8}$$

The limit of the ratio $Y_{12}'(z)/Y_{12}(z)$ as $z \to 0$ is found by considering the equation for $Y_{12}(z)$, which in general form can be written as

$$Y'' - [p + f(z)]Y = 0, \tag{9}$$

where $f(z) \to 0$ as $z \to 0$. The substitution $z = e^{-2x}$ and $Y = ue^{-x}$ leads to

$$u'' - u\{1 + 4e^{-4x}[p + f(e^{-2x})]\} = 0; \tag{10}$$

as $x \to \infty$, $u'/u \to -1$ ([2]), whence $Y_{12}'/Y_{12} = u'/u - 1 = -2$.

If we denote by L^{-1} the inversion operation with respect to all variables and take into account that the surface concentration of the deposited impurity $\sigma(x, y)$ is the integral with respect to t from 0 to ∞ of the impurity flux, then, to estimate the error in computing σ, it is necessary to carry out computations that are rather cumbersome in volume:

$$\begin{aligned}\sigma(x,y) &= \int_0^\infty L^{-1}(\Pi_0)\,dt \simeq \int_0^\infty L^{-1}(\Pi_0|_{H=0})\,dt + H\int_0^\infty L^{-1}\left(\frac{\partial \Pi_0}{\partial H}\bigg|_{H=0}\right)dt \\ &= \sigma|_{H=0}\left\{1 + H\int_0^\infty L^{-1}\left(\frac{\partial \Pi_0}{\partial H}\bigg|_{H=0}\right)dt\bigg/\int_0^\infty L^{-1}(\Pi_0|_{H=0})\,dt\right\}.\end{aligned} \tag{11}$$

After passing in (11) to the original variables, in the integration with respect to t one encounters integrals of the form

$$I = \int_0^\infty \varphi(t) e^{-a/t - bt}, \tag{12}$$

where $a = x^2/4k_x + y^2/4k_y$; $b = w^2/2k_1$, which were estimated as follows: after the substitution $t = \tau\sqrt{a/b}$, we obtain

$$I = 2K_1(2\sqrt{ab})\sqrt{\frac{a}{b}} \int_0^\infty \varphi\left(\tau\sqrt{\frac{a}{b}}\right) \left\{\frac{e^{-\sqrt{ab}(\tau+1/\tau)}}{2K_1(2\sqrt{ab})}\right\} d\tau,$$

where the braces enclose a function which, as $\sqrt{ab} \to \infty$, approximates the Dirac δ-function $\delta(\tau - 1)$. Expanding it in a symbolic series ([3]), we have, for large values of $\sqrt{ab}$,

$$\frac{e^{-\sqrt{ab}(\tau+1/\tau)}}{2K_1(2\sqrt{ab})} \simeq \delta(\tau - 1) + \frac{1}{\sqrt{ab}} \frac{\tau^3}{(1+\tau)^2} \delta''(\tau - 1),$$

whence

$$I \simeq 2K_1(2\sqrt{ab})\sqrt{\frac{a}{b}}\,\varphi\left(\sqrt{\frac{a}{b}}\right) \left[1 + \frac{1}{\sqrt{ab}} \left(\frac{3}{8} + \frac{\varphi'_\tau}{\varphi} + \frac{1}{4}\frac{\varphi''_\tau}{\varphi}\right)_{\tau=1}\right] \tag{13}$$

($K_1(x)$ is the Bessel function).

As a result, from (11) we have

$$\sigma(x, y, h, H, k_1, k_2) \simeq$$

$$\simeq \sigma(x, y, h, 0, k_1) \left\{1 - \frac{H}{2h}\left(\frac{k_1}{k_2} - 1\right) \frac{xw/\sqrt{k_1 k_x}}{\sqrt{1 + x^2 k_1/h^2 k_x + y^2 k_1/h^2 k_y}}\right\}, \tag{14}$$

where the magnitude of the velocity w is bounded below by the relation

$$w > \frac{\sqrt{k_{xk}1}}{x}. \tag{15}$$

The estimate obtained becomes ineffective in the case $k_2 \sim 0$, to the investigation of which we now turn.

Here we shall obtain an estimate of the error that will be admitted if turbulent scattering in the vertical direction in the ground layer is not taken into account at all when computing the surface concentration; this is evidently connected with integration with respect to all t of the original of the expression

$$\Pi_0 \simeq \Pi_0\big|_{k_2=0} + k_2 \frac{\partial \Pi_0}{\partial k_2}\Big|_{k_2=0} =$$

$$= -C_1 e^{-wH/2k_1} \frac{w\sqrt{w^2+4pk_1}\, A_1(0) \exp\left[-\frac{Hp}{w} + \frac{1}{H}\int_0^H u(z)\, dz \cdot p_\xi \frac{H}{w\sqrt{k_x}}\right]}{wY_{12}(H) - \sqrt{w^2+4pk_1}\, Y_{12}'(H)} \times$$

$$\times \left[1 + k_2 \frac{p_\xi M(k+1) u(H)}{w^2\sqrt{k_x}}\right]. \tag{16}$$

The constant $M \geq 2$ appears in expanding the functions $F_{21}(\xi_2)$ and $F_{22}(\xi_2)$ in small values of k_2, after replacing in them the polynomials $\chi_n(\xi_2)$ by the corresponding majorizing sequence

$$\overline{\chi}(z) = \frac{1}{n!(k+1)^n} \left[\frac{z+N}{k_2}\sqrt{w^2+4pk_2} + M(k+1)\prod_{s=2}^{n} \frac{s(k+1)}{s(k+1)-1}\right]^{n(k+1)}, \tag{17}$$

where k is the degree of the polynomial by which the variation of wind velocity with height is specified:

$$u(z) = \sum_{m=0}^{k} b_m z^m.$$

Passing in (16) to the original and integrating with respect to t from 0 to ∞, we obtain the desired expression, in which the principal term does not contain k_2, while the second term makes it possible to estimate the required error:

$$\sigma(x, y, H, h, k_1, k_2) = \sigma(x, y, H, h, k_1, 0) \left[1 - \frac{M(k+1)}{2} \frac{k_2 u(H)}{w\sqrt{k_1 k_x}}\right] \tag{18}$$

under condition (15).

In conclusion, we note that both estimates were obtained by specifying the variation of wind velocity with height by means of a polynomial of arbitrary degree k in z, but with the condition that in the series of originals for expressions (16) and (11) a finite number of terms is used; this is possible, since the convergence of the resulting series has been proved in (1). For the same reason there is no need to analyze the subsequent terms in the expansions (14) and (18).

Institute of Applied Geophysics
Academy of Sciences of the USSR

Received
30 V 1960

REFERENCES

1. O. S. Berlyand, *Inzh.-fiz. zhurn.*, **2**, No. 5 (1959).
2. R. Bellman, *Theory of the Stability of Solutions of Differential Equations*, IL, 1954.
3. A. Ya. Pressman, O. S. Berlyand, *DAN*, **126**, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.