MATHEMATICS

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Abstract

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MATHEMATICS

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ON A VERY SHORT EXPONENTIAL RATIONAL TRIGONOMETRIC SUM

(Presented by Academician I. M. Vinogradov on 15 IV 1960)

The present paper is an analogue of the work of M. P. Mineev ([1]) and is a new application of A. A. Markov' s method of moments* to number-theoretic problems.

Let $g \geqslant 2$ be a natural number. Let p be a prime number, and $h = h(p)$ some integer-valued function; $h \to \infty$ as $p \to \infty$; $h(p) \leqslant \frac{1}{2}\frac{\log p}{\log g}$. Let $\lambda > 0$ be a constant. Denote by $N_p(\lambda)$ the number of integers a, $0 \leqslant a \leqslant p-1$, for which

$$\left|\sum_{x=0}^{h} \exp\left[2\pi i\,\frac{ag^x}{p}\right]\right| < \lambda\sqrt{h}.$$

Theorem. As $p \to \infty$,

$$\lim_{p\to\infty} \frac{N_p(\lambda)}{p} = 1 - e^{-\lambda^2}.$$

Proof. For fixed p and, consequently, h, consider the random variable ξ_p, taking the values

$$\frac{1}{h}\left|\sum_{x=0}^{h} \exp\left[2\pi i\,\frac{ag^x}{p}\right]\right|^2, \qquad a = 0, 1, \dots, p-1,$$

with probability equal to $1/p$. The distribution function of this random variable is $N_p(\lambda^2)/p$.

Let us compute the r-th moment of this distribution function:

$$\frac{1}{p}\sum_{a=0}^{p-1} \frac{1}{h^r}\left|\sum_{x=0}^{h} \exp\left[2\pi i\,\frac{ag^x}{p}\right]\right|^{2r} =$$

$$= \frac{1}{ph^r} \sum_{x_1=0}^{h} \cdots \sum_{x_r=0}^{h} \sum_{y_1=0}^{h} \cdots \sum_{y_r=0}^{h} \exp\left[2\pi i \, \frac{a\left(g^{x_1} + \cdots + g^{x_r} - g^{y_1} - \cdots - g^{y_r}\right)}{p}\right] = \frac{1}{h^r} M_r(p),$$

where $M_r(p)$ is the number of solutions of the congruence

$$g^{x_1} + \cdots + g^{x_r} \equiv g^{y_1} + \cdots + g^{y_r} \pmod p \tag{1}$$

in the integers $0 \leqslant x_i, y_i \leqslant h$, $i = 1, 2, \ldots, r$. But since $0 \leqslant x_i, y_i \leqslant h \leqslant \frac{1}{2} \frac{\log p}{\log g}$, we have $r \leqslant g^{x_1} + \cdots + g^{x_r} \leqslant r\sqrt{p}$.

* An application of the method of moments similar in idea is found in the works ([2], [3]).

If $p > r^2$, then

$$0 < g^{x_1} + \ldots + g^{x_r} < p,$$

and the congruence of the two sides of (1) means equality. Therefore $M_r(p)$ is equal to the number of solutions of the equation

$$g^{x_1} + \ldots + g^{x_r} = g^{y_1} + \ldots + g^{y_r} \tag{2}$$

in numbers $0 \leq x_i, y_i \leq h$.

Let r be fixed, $p \to \infty$ (and hence $h \to \infty$). The number of solutions of equation (2) is expressed by the formula

$$M_r(p) = r!\, h^r + O\left(h^{r-1}\right)$$

(a more general assertion was proved in paper [4]).

Thus,

$$\lim_{p\to\infty} \frac{1}{p} \sum_{a=0}^{p-1} \frac{1}{h^r} \left| \sum_{x=0}^{h} \exp\left[2\pi i \frac{a g^x}{p}\right] \right|^{2r} = r!$$

Applying the second limit theorem of probability theory in the way this was done in paper [1], we obtain

$$\lim_{p\to\infty} \frac{N_p(\lambda^2)}{p} = 1 - e^{-\lambda},$$

which is what was required to prove.

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REFERENCES

[1] M. P. Mineev, *UMN*, **14**, no. 3, 169 (1959).
[2] H. Davenport, P. Erdős, *Publ. Math.*, **2**, 252 (1952).
[3] I. P. Kubilius, Yu. V. Linnik, *Izv. Higher Educational Institutions, Mathematics*, no. 6, 88 (1959).
[4] M. P. Mineev, *Matem. sbornik*, **46** (88), 4, 451 (1958).

Note: Figure translations are in progress. See original paper for figures.

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