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Abstract

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THE RING OF QUOTIENTS WITH RESPECT TO A PRIME IDEAL

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Let R be an arbitrary associative ring and let S be some multiplicatively closed system of its elements, possibly containing zero divisors of the ring R, but not zero. If the ring R has a generalized left ring of quotients with respect to the system S, then by I we shall denote the two-sided ideal of the ring R that is the intersection of all its S-prime ideals, and by φ the homomorphic mapping of the ring R into the ring $R_{(S)}$ with kernel I (^{1}).

Lemma 1. *For any finite number of elements* $x_i \in R_{(S)}$ *there exists an element* $\bar{s} \in S$ *such that* $\varphi(\bar{s})x_i = \varphi(\bar{r}_i)$, *where* $\bar{r}_i \in R$.

Proof. $x_i = [\varphi(s_i)]^{-1}(r_i)$, where $s_i \in S,\ r_i \in R,\ i = 1, \dots, n$. For s_1 and any $s_0 \in S$, by the definition of the ideal I, there are elements $s' \in S$ and $r' \in R$ such that $r's_1 - s's_0 \in I$, i.e. $\varphi(r')\varphi(s_1) = \varphi(s')\varphi(s_0) = \varphi(\bar{s})$, where $\bar{s} = s's_0 \in S$. Suppose that for $i = 1, \dots, n{-}1$ there exists an element $\bar{s}' \in S$ such that $\varphi(\bar{s}') = \varphi(\bar{r}_i')\varphi(s_i)$, where $\bar{r}_i' \in R$. For $\bar{s}'$ and s_n there are elements $s'' \in S$ and $r'' \in R$ such that $\varphi(r'')\varphi(s_n) = \varphi(s'')\varphi(\bar{s}')$. Then $\varphi(\bar{s}) = \varphi(s'')\varphi(\bar{s}') = \varphi(\bar{r}_i)\varphi(s_i)$, where $\bar{r}_i = s''\bar{r}_i',\ i = 1, \dots, n-1$. Thus $\varphi(\bar{s}) = \varphi(\bar{r}_i)\varphi(s_i)$, where $\bar{r}_n = r'',\ i = 1, \dots, n$. Then $\varphi(\bar{r}_i)\varphi(s_i)x_i = \varphi(\bar{r}_i)\varphi(r_i)$, i.e. $\varphi(\bar{s})x_i = \varphi(\bar{r}_i')$, where $\bar{r}_i' \in R$. The lemma is proved.

If L is a left ideal of the ring R, then by

$$L^e = R_{(S)}\varphi(L)$$

we shall denote the left ideal of the ring $R_{(S)}$ generated by the set $\varphi(L)$. If $L_{(S)}$ is a left ideal of the ring $R_{(S)}$, then by

$$L_{(S)}^c = \varphi^{-1}(L_{(S)} \cap \varphi(R))$$

we shall denote the left ideal of the ring R that is the complete inverse image of $L_{(S)} \cap \varphi(R)$.

Lemma 2. *The ideal* L^{ec} *consists of those elements* $b \in R$ *for which there exist elements* $s \in S,\ l \in L$ *such that* $sb - l \in I$.

Proof. Let $b \in R,\ b \in L^{ec} = (R_{(S)}\varphi(L) \cap \varphi(R))$. Then $\varphi(b) \in R_{(S)}\varphi(L) = L^e$, and

$$\varphi(b) = \sum_{i=1}^{n} r_s^{(i)} \varphi(l_i),$$

where $r_s^{(i)} \in R_{(S)},\ l_i \in L$. There exists (Lemma 1) an element $s \in S$ such that $\varphi(s) r_s^{(i)} = \varphi(r_i),\ r_i \in R$. Then $\varphi(s)\varphi(b) = \varphi(l)$, where $l \in L$. Consequently, $\varphi(sb - l) = 0$ and $sb - l \in I$. If for $b \in R$ there are elements $s \in S$ and $l \in L$ such that $sb - l \in I$, then $\varphi(b) = [\varphi(s)]^{-1}\varphi(l) \in R_{(S)}\varphi(L) = L^e$. But then $b \in \varphi^{-1}(L^e \cap \varphi(R)) = L^{ec}$, and the lemma is proved.

Lemma 3. *$L^e \neq R_{(S)}$ if and only if, for any elements $s \in S,\ l \in L,\ l - s \notin I$.*

Proof. If $L^e = R_{(S)}\varphi(L) = R_{(S)}$, then $L^{ec} = \varphi^{-1}(L^e \cap \varphi(R)) = R$. By Lemma 2, for any element $r \in R$ there exist elements $s \in S$ and $l \in L$ such that $sr - l \in I$. If $r = s' \in S$, then $s's - l \in I$, where $ss' \in S$. If $L^e \neq R_{(S)}$, then $R_{(S)}\varphi(L) \neq R_{(S)}$. If $l - s \in I$, then $\varphi(l) = \varphi(s)$ and

$R_{(S)}\varphi(L) = L^e \supseteq R_{(S)}\varphi(S) \ni 1$, i.e. $L^e = R_{(S)}$, which is impossible. Hence, $l - s \notin I$. The lemma is proved.

A two-sided ideal P of a ring R will be called **prime** if from $r_1 r_2 \in P$, where $r_1 \in R,\ r_2 \in R$, it follows that $r_1 \in P$ or $r_2 \in P$. If R is an arbitrary associative ring and P is its proper prime ideal, then denote $S = R - P$. S is a multiplicatively closed system of elements of the ring R without zero. If the ring $R_{(S)}$ exists, then we denote it by $R_{(P)}$ and call it the generalized left ring of fractions of the ring R with respect to the prime ideal P.

Theorem 1. *If the ring R has the ring $R_{(P)}$, then:*

1) *the elements of the ring $R_{(P)}$ not belonging to the ideal P^e have two-sided inverses in $R_{(P)}$;*
2) *the elements of the ideal P^e are neither left nor right divisors of unity;*
3) *every proper ideal of the ring $R_{(P)}$ is contained in the ideal P^e.*

Proof. Let $x \in R_{(P)},\ x \notin P^e$. Then $x = [\varphi(s)]^{-1}\varphi(s_1)$, where $s, s_1 \in S$, and $x[\varphi(s_1)]^{-1}\varphi(s) = [\varphi(s_1)]^{-1}\varphi(s)x = 1$. Hence x has a two-sided inverse.

Let y be an arbitrary element of P^e. If $y'y = 1$, where $y' \in R_{(P)}$, then $P^e = R_{(P)}$. If $p \in P,\ s \in S$ and $p - s \in I$, then $p - s \in P$, since $I \subseteq P$ and $s \in P$, which is impossible. Therefore, by Lemma 3, $P^e \neq R_{(P)}$, and y is not a right divisor of unity. If $yy' = 1$, where $y' \in R_{(P)}$, then y' is a right divisor of unity, i.e. $y' \notin P^e$. Consequently, there is an element $y'' \in R_{(P)}$ such that $y'y'' = y''y' = 1$. Then $(y - y'')y' = 0$ and $y = y''$, i.e. $y'y = 1$ and y is a right divisor of unity, which is impossible. Consequently, y is not a left divisor of unity.

The third assertion of the theorem follows directly from the second. The theorem is proved.

An associative ring R will be called **local** (on the left, on the right, two-sided) if in the ring R there exists a unique (left, right, two-sided) maximal ideal.

Corollary. The ring $R_{(P)}$ is local on the left.

A left ideal L of the ring R will be called **contracted** if $L = L^{ec}$.

Lemma 4. *An ideal L is contracted if and only if from $sr - l \in I$, where $s \in S$, $r \in R$, $l \in L$, it follows that $r \in L$.*

Proof. Suppose that from $sr - l \in I$ it follows that $r \in L$, and let $b \in R$, $b \in L^{ec}$. Then (Lemma 2) there exist elements $s' \in S$ and $l' \in L$ such that $s'b - l' \in I$. By assumption, $b \in L$. Thus $L^{ec} \subseteq L$. On the other hand, $L \subseteq L^{ec}$, i.e. $L = L^{ec}$.

Now suppose that $L = L^{ec}$ and $sr - l \in I$. Then $\varphi(s)\varphi(r) = \varphi(l)$ and $\varphi(r) = [\varphi(s)]^{-1}\varphi(l) \in R_{(S)}\varphi(L) = L^e$. Hence $r \in \varphi^{-1}(L^e \cap \varphi(R)) = L^{ec} = L$. The lemma is proved.

If R, R_1 and R_2 are associative rings and φ_1, φ_2 are homomorphic mappings of the ring R into R_1 and R_2, respectively, then we shall say that $R_1 \cong R_2$ over R if there exists an isomorphism of the rings R_1 and R_2 under which the images of one and the same element of the ring R correspond to one another.

If, alongside the left generalized ring of fractions $R_{(S)}$ of the ring R with respect to the system S, there exists the right generalized ring of fractions $\overline{R}_{(S)}$ with respect to the same system S (it can be defined analogously to the way the ring $R_{(S)}$ is defined in [1]), then it is easy to verify that $R_{(S)} \cong \overline{R}_{(S)}$ over R if and only if the kernel I of the mapping φ of the ring R into the ring $R_{(S)}$ coincides with the kernel $\overline{I}$ of the mapping $\overline{\varphi}$ of the ring R into the ring $\overline{R}_{(S)}$.

Lemma 5. *If the ring R has the rings $R_{(S)}$, $\overline{R}_{(S)}$ and $R_{(S)} \cong \overline{R}_{(S)}$ over R; A is a two-sided ideal of the ring R which is contracted as a right ideal, then the left ideal A^e is two-sided in the ring $R_{(S)}$.*

Proof. $A^e = R_{(S)}\varphi(A) = \{\sum r_s^{(a)}\varphi(a_a)\}$, where $r_s^{(a)} \in R_{(S)}$, $a_a \in A$. Let $a^e \in A^e$ and $x \in R_{(S)}$. By the hypothesis of the theorem, for any elements $r \in R$ and $s \in S$ there exist elements $r' \in R$ and $s' \in S$ such that $rs' - sr' \in \bar{I} = I$. If $x = [\varphi(s)]^{-1}\varphi(r)$, then $\varphi(r)\varphi(s') = \varphi(s)\varphi(r')$ and $x\varphi(s') = [\varphi(s)]^{-1}\varphi(r)\varphi(s') = [\varphi(s)]^{-1}\varphi(s)\varphi(r') = \varphi(r')$, i.e. $x = \varphi(r')[\varphi(s')]^{-1}$. Then

$$a^e x = [r_s^{(1)}\varphi(a_1) + \cdots + r_s^{(n)}\varphi(a_n)]\varphi(r')[\varphi(s')]^{-1}.$$

For the elements $r_s^{(i)}$ there is (Lemma 1) an element $\bar{s} \in S$ such that $\varphi(\bar{s})r_s^{(i)} = \varphi(\bar{r}_i)$, where $\bar{r}_i \in R$. Then

$$\varphi(\bar{s})a^e x = [\varphi(\bar{r}_1 a_1 r') + \cdots + \varphi(\bar{r}'_{na_{nr}})][\varphi(s')]^{-1} = [\varphi(a'_1) + \cdots + \varphi(a'_n)][\varphi(s')]^{-1},$$

where $\bar{r}'_{ia_{ir}} = a'_i \in A$. Hence $\varphi(\bar{s})a^{ex} = \varphi(a)[\varphi(s')]^{-1}$, where $a \in A$, and

$$a^{ex} = [\varphi(\bar{s})]^{-1}\varphi(a)[\varphi(s')]^{-1}.$$

But $\varphi(a)[\varphi(s')]^{-1} \in R_{(S)}$, i.e. $\varphi(a)[\varphi(s')]^{-1} = [\varphi(s_0)]^{-1}\varphi(r_0)$, $s_0 \in S$, $r_0 \in R$. Then $\varphi(r_0)\varphi(s') = \varphi(s_0)\varphi(a) = \varphi(a')$, where $a' \in A$. Consequently, $r_0 s' - a' \in I$, and since A is a right contracted ideal, $r_0 \in A$ (by the analogue of Lemma 4 for the ring $\bar{R}_{(S)}$). Hence $a^{ex} = [\varphi(\bar{s})]^{-1}[\varphi(s_0)]^{-1}\varphi(r_0) \in R_{(S)}\varphi(A) = A^e$. The lemma is proved.

Theorem 2. *If the ring R has rings $R_{(P)}$ and $\bar{R}_{(P)}$, and $R_{(P)} \cong \bar{R}_{(P)}$ over R, then the ring $R_{(P)}$ ($\bar{R}_{(P)}$) is local.*

Proof. If $sr - p \in I$, where $s \in S$, $p \in P$, and $r \in R$, then $sr - p \in P$, since $I \subset P$. Then $sr \in P$ and, since $s \notin P$, it follows that $r \in P$. Thus the ideal P is right contracted. It is checked analogously that the ideal P is left contracted. Hence, by Lemma 5, it follows that the left ideal $P^e = R_{(P)}\varphi(P)$ is two-sided in the ring $R_{(P)}$, and by the analogue of this lemma for the ring $\bar{R}_{(P)}$, the right ideal $P^e = \varphi(P)\bar{R}_{(P)}$ is two-sided in the ring $\bar{R}_{(P)}$. After this, Theorem 2 is a consequence of Theorem 1 and of its analogue for the ring $\bar{R}_{(P)}$.

If R is a commutative integral domain, then the results of Theorems 1 and 2 coincide with the results set forth in the book of Hodge and Pedoe ([2], p. 55).

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[2] V. Hodge, D. Pedoe, *Methods of Algebraic Geometry*, 3, IL, 1955.

Note: Figure translations are in progress. See original paper for figures.

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