Hydromechanics

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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On the Plane Flow of an Infinitely Conducting Fluid with Almost Parallel Vectors of the Magnetic Field and Velocity

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§ 1. If the vectors of the magnetic field **H** and velocity **V** are parallel at infinity, then in an infinitely conducting fluid they are parallel throughout the entire flow (see, for example, ([1])). If such a flow passes around a body in which there are no sources of magnetic field, then inside the body the field is equal to zero, and at the boundary of the body and the fluid a discontinuity of the magnetic field occurs, i.e., a surface current flows. The force acting on this current is perpendicular to the surface of the body and is perceived by it as magnetic pressure.

Fig. 1

If, however, in a plane flow $\mathbf{H} \neq \mathbf{V}$ at infinity, then they are not parallel throughout the flow, since $[\mathbf{V}, \mathbf{H}] = \text{const}$ throughout the entire flow.* On the surface of the body, we obviously have:

$$V_t H_n = U_\infty H_{y\infty}, \tag{1}$$

where V_t and H_n are, respectively, the tangential and normal components of the velocity and field vectors; U_∞ is the velocity of the oncoming flow, directed along the x axis; $H_{y\infty}$ is the component of the magnetic field along the y axis (see Fig. 1).

From (1) two important consequences immediately follow:

1. $H_n \neq 0$ on the surface of the body. Consequently, on the surface of a dielectric body there can be no surface currents, since otherwise the force acting on this current would be parallel to the surface, which is impossible in an ideal fluid.
2. When approaching a critical point ($V_t \to 0$), the field $H_n \to \infty$.

In the present paper a new type of boundary layer will be considered, connected with the first of the facts indicated and arising on the surface of bodies when the velocity and field vectors are almost parallel (i.e., for small $\varepsilon = H_{y\infty}/H_{x\infty}$) in an ideal infinitely conducting fluid.

§ 2. To clarify the nature of the phenomenon, let us consider the plane flow of an ideal infinitely conducting incompressible fluid with almost parallel vectors of field and velocity. On the surface of the body $H_n = O(\varepsilon)$. Consequently, inside the body at the surface $H_t = O(\varepsilon)$. Since, according to what was said above, there can be no discontinuity of the field at the surface, then also inside the fluid at the surface $H_t = O(\varepsilon)$. On the other hand,

* It is assumed that an electric field perpendicular to the plane of the drawing is imposed on the flow, compensating the induction field.

it is clear that the entire flow field (with the exception of special regions) must differ little from the flow for $\varepsilon = 0$. In the latter case, however, on the surface of the body in the fluid $H_t = O(1)$. Consequently, the discontinuity of the field at the surface, which existed for $\varepsilon = 0$, is smeared out for $\varepsilon \neq 0$ into a boundary layer of thickness of order ε, in which the field changes from $H_t = O(\varepsilon)$ at the surface to $H_t = O(1)$ at the boundary of the layer. Thus the whole flow can be divided into two principal regions: a flow differing little from the flow for $\varepsilon = 0$ (with parallel vectors of the field and of the velocity), and a flow in the boundary layer (Fig. 1). Regions near critical points are also special.

The equations describing the flow have the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0; \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{H_y}{4\pi\rho}\left[\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right]; \tag{3a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{H_x}{4\pi\rho}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]; \tag{3b}$$

$$uH_y - vH_x = H_{y\infty}U_\infty. \tag{4}$$

Carrying out the estimates customary for flows in boundary layers, we obtain from (3b) that $p + H_x^2/8\pi$ is constant across the layer. Further, for the simplest case, when outside the boundary layer $\partial p/\partial x = 0$, equation (3a) takes the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{4\pi\rho}\left(H_x\frac{\partial H_x}{\partial x} + H_y\frac{\partial H_x}{\partial y}\right), \tag{5}$$

where now $x,\ y$ are, respectively, the coordinate along and normal to the surface. Equations (2) and (4) remain unchanged. Equations (2) make it possible to introduce functions ψ and χ such that

$$u = \frac{\partial \psi}{\partial y}; \qquad v = -\frac{\partial \psi}{\partial x}; \qquad H_x = \frac{\partial \chi}{\partial y}; \qquad H_y = -\frac{\partial \psi}{\partial x}. \tag{6}$$

Differentiating (4) with respect to x and y and using (2), we obtain two more equations:

$$u\frac{\partial H_y}{\partial x} + v\frac{\partial H_y}{\partial y} = H_x\frac{\partial v}{\partial x} + H_y\frac{\partial v}{\partial y};$$

$$H_x\frac{\partial u}{\partial x} + H_y\frac{\partial u}{\partial y} = u\frac{\partial H_x}{\partial x} + v\frac{\partial H_x}{\partial y}. \tag{7}$$

We pass in equations (5) and (7) to the independent variables ψ and χ; we have:

$$\frac{1}{4\pi\rho}\frac{\partial H_x}{\partial \psi} + \frac{\partial u}{\partial \chi} = 0; \qquad \frac{\partial H_x}{\partial \chi} + \frac{\partial u}{\partial \psi} = 0; \tag{8}$$

$$\frac{\partial H_y}{\partial \chi} + \frac{\partial v}{\partial \psi} = 0. \tag{9}$$

Equations (8) are of hyperbolic type and possess two families of characteristics:

$$\psi'_{1,2} = \left.\frac{d\psi}{d\chi}\right|_{1,2} = \pm\frac{1}{\sqrt{4\pi\rho}}. \tag{10}$$

Along the characteristics the following relations are satisfied:

$$u' + \psi'_{1,2}H'_x = 0. \tag{11}$$

Equation (9) will obviously be satisfied if one introduces a function Φ such that

$$v = \frac{\partial \Phi}{\partial \chi}; \qquad H_y = -\frac{\partial \Phi}{\partial \psi}.$$

Then from (4) we obtain

$$u\frac{\partial \Phi}{\partial \psi} + H_x\frac{\partial \Phi}{\partial \chi} = -H_{y\infty}U_\infty. \tag{12}$$

It is clear that on one side of the critical point the field enters the body, while on the other it leaves it (Fig. 1). The boundary of the boundary layer will be the lines $\chi = \text{const}$ passing through the critical points. Assigning to the field line passing through the front critical point the value $\chi = 0$, one can formulate the boundary-value problem for the boundary layer in the plane $(\chi; \psi)$ as follows.

For the outgoing layer (Fig. 1), on the line $\chi = 0$ the functions $u = u_0(\psi)$ and $H_x = H_0(\psi)$ are prescribed. At $\psi = 0$, to within higher-order terms, $H_x = 0$ and $v = \partial\Phi/\partial\chi = 0$, or $\Phi = 0$. Similarly, for the incoming layer, $u(\psi)$ and $H_x(\psi)$ are prescribed on the field line $\chi = C$, passing through the rear critical point.

The solution of system (8) can obviously be readily expressed in explicit form in terms of the functions $H_0(\psi)$ and $u_0(\psi)$. The solution of equation (12), after u and H_x have been found, also presents no fundamental difficulties. The transition to the physical plane is carried out by the formulas

$$dx = \frac{1}{H_{y\infty}U_\infty}(u\,d\chi - H_x\,d\psi); \qquad dy = \frac{1}{H_{y\infty}U_\infty}(v\,d\chi - H_y\,d\psi). \tag{13}$$

The relation between ψ and χ along the boundary line $\gamma = \text{const}$ is determined by the relation $d\psi = -\frac{1}{H_0(\chi)}\,d\chi$. Here $H_0(\chi)$ is the value of the function obtained from the solution of the exterior problem. Over the greater part of the boundary layer, with the required accuracy, it is evidently possible to take $H_0(\chi)$ and $u_0(\chi)$ from the solution for $\varepsilon = 0$. However, owing to the above-mentioned peculiarity near the critical points, the flow cannot be regarded as differing little from the flow for $\varepsilon = 0$. Therefore this region requires special consideration.

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CITED LITERATURE

1. M. N. Kogan, *Prikl. matem. i mekh.*, **23**, No. 1, 70 (1959).

2. L. D. Landau, E. M. Lifshitz, *Electrodynamics of Continuous Media*, Moscow, 1957.

Note: Figure translations are in progress. See original paper for figures.