Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1960. Vol. 131, No. 3

MATHEMATICS

Yu. M. BEREZANSKII

SOME EXAMPLES OF "NONCLASSICAL" BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev on 27 XI 1959)

This note gives proofs of energy inequalities and thereby establishes the existence of weak solutions of boundary-value problems for certain classes of partial differential equations that have been little studied from this point of view. The method for obtaining the inequalities is a development of the technique used in ([1]) (example 4).

1°. Let us consider, in a bounded domain G of the n-dimensional space E_n, a linear differential expression $\mathcal{L}$ of order r with complex-valued sufficiently smooth coefficients; $\mathcal{L}^+$ is the formally adjoint expression, Γ is the boundary of G. As in ([1]), we introduce boundary conditions: by functions satisfying certain boundary conditions (bc) we shall mean a subspace of the Sobolev space W_2^r, containing $\overset{\circ}{W}{}_2^r$—the closure in the metric of finit supported (i.e. vanishing in a strip near Γ) functions from W_2^r. We denote this subspace by $W_2^r(\mathrm{bc})$. The subspace $W_2^r(\mathrm{bc})^+$, corresponding to the adjoint boundary conditions, is introduced as the set of functions v from W_2^r such that $(\mathcal{L}[u], v)_0 = (u, \mathcal{L}^+[v])_0$ for every $u \in W_2^r(\mathrm{bc})$ ($(f, g)_0$ is the scalar product in L_2). It is required of the boundary conditions (bc) that $(\mathrm{bc})^{++} = (\mathrm{bc})$.

Let us introduce a negative norm (see ([2–4])). Namely, suppose that some linear set H^+, dense in L_2, forms, with respect to a new scalar product $(u, v)_+$, a Hilbert space, and that $\|u\|_0 \le \|u\|_+$. It is not difficult to show that every linear continuous functional $l(u)$ in H^+ can be represented in the form $l(u) = (\alpha, u)_0$, where α is an element of a certain Hilbert space H^-—a space with negative norm. Clearly, H^- consists of generalized functions of a definite type, $H^- \supset L_2$. In what follows we shall consider only such positive norms $\|u\|_+$ for which $H^+ \supset W_2^r$.

2°. Let H^+ and $H^{+,*}$ be two spaces with positive norms. With the aid of the Hahn–Banach theorem it is easily shown that the validity of the inequality

$$\|\mathcal{L}^+[v]\|_0 \geq c\|v\|_{+,*} \qquad (c > 0,\ v \in W_2^r(\mathrm{bc})^+) \tag{1}$$

implies the solvability of the boundary-value problem $\mathcal{L}[u] = f \in H^{-,*}$, $u \in (\mathrm{bc})$, in the weak sense, i.e. it implies the existence of such a $u \in L_2$ that $(u, \mathcal{L}^+[v])_0 = (f, v)_0$ for all $v \in W_2^r(\mathrm{bc})^+$. We shall call the boundary conditions (bc) correct (cf. ([5]), Ch. 7) if, together with inequality (1), the inequality

$$\|\mathcal{L}[u]\|_0 \geq c\|u\|_+ \qquad (u \in W_2^r(\mathrm{bc})) \tag{2}$$

is satisfied.

If it turns out that the weak solution belongs to W_2^r, then it also belongs to $W_2^r(\mathrm{bc})$, $f \in L_2$, and $\mathcal{L}[u] = f$. Inequality (2) shows that the considered—

problem has a unique solution, continuously depending on the right-hand side, i.e. it is well-posed. In the general case, well-posedness of the boundary conditions means only the existence of a weak solution and the uniqueness and continuous dependence on f of a smooth solution. We shall not dwell here on questions of smoothness of weak solutions.

3°. The derivation of inequalities (1)–(2) is simple in the case of definiteness of the polylinear form corresponding to $\mathcal{L}$. Below we shall outline one way of deriving them without this assumption, which in a number of cases leads to the goal.

Let

$$\mathcal{L}[u] = \sum_{|\alpha| \leq r} a_\alpha D^\alpha u$$

$$\left(D^\alpha = i^{-|\alpha|} D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_j = \frac{\partial}{\partial x_j}, \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \alpha_1 + \dots + \alpha_n\right) \tag{3}$$

be an expression with constant coefficients, $\mathcal{L}(\xi) = \sum a_\alpha \xi^\alpha$ $(\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n})$.

Set $\mathcal{L}^{(\beta)}(\xi) = D^\beta \mathcal{L}(\xi)$, and by $\mathcal{L}^{(\beta)}[u]$ denote the corresponding differential expression. Hörmander showed ([6]) that for functions finite in G the inequality

$$\|\mathcal{L}[u]\|_0 \geq c\|\mathcal{L}^{(\beta)}[u]\|_0 \tag{4}$$

holds for any differentiation D^β. Taking $\sum \|\mathcal{L}^{(\beta)}[u]\|_0^2$ as $\|u\|_r^2$, we obtain from (4) an inequality of type (2).

Let us now show how to obtain inequalities of type (4) for classes of functions broader than the finite ones. First of all note that (4) on $\mathring{W}_2^r$ can be obtained simply not by means of the Fourier transform, as was done in ([6]), but by ordinary integration by parts (for $r \leq 2$ this was noted by the author; in the general case–by L. P. Nizhnik). The proof reduces to the fact that the integral $\int_G x_k \mathcal{L}[u]\mathcal{L}^{(k)}[u]\,dx$, where $\mathcal{L}^{(k)}$ is constructed with respect to $\partial/\partial\xi_k$, is transformed by integration by parts into the integral $\int_G x_k \mathcal{L}^+[u]\mathcal{L}^{(k)+}[u]\,dx$. In the resulting identity there necessarily appears the term $\int_G |\mathcal{L}^{(k)}[u]|^2\,dx$, which, if induction on the order of the expression $\mathcal{L}$ is applied, makes it possible to obtain the estimate (4) with $\mathcal{L}^{(\beta)} = \mathcal{L}^{(k)}$, and hence also with any $\mathcal{L}^{(\beta)}$.

Now let u not be finite. We perform the mentioned transformation, adding everywhere integrals over Γ. In order to obtain (4), we shall impose such boundary conditions on u that all these integrals vanish or become definite of the required sign. As a result we obtain the desired boundary conditions, depending both on $\mathcal{L}$ and on Γ. We note that such a procedure usually leads to rigid conditions on $u|_\Gamma$; for each separate class of equations it should be modernized. As will be seen below, it can also be applied to some equations with variable coefficients.

After these general remarks let us proceed to the consideration of concrete classes of equations. We shall restrict ourselves to the case $r \leq 2$ and to three types of equations.

4°. Let $\mathcal{L}$ be a formally self-adjoint expression of the form (3), $r = 2$. Suppose that in the matrix $\|a_{jk}\|$ of the coefficients standing at $D_{jD_{ku}}$ in $\mathcal{L}$, $a_{11} \neq 0$ and $a_{1j} = a_{j1} = 0$ for $j \neq 1$. Let G be a cylinder with generatrix parallel to Ox_1, and with bases Γ_l of the form $x_1 = 0$ and Γ_p of the form $x_1 = c > 0$; Γ_b is its lateral surface. Applying the procedure described in item 3°–

procedure, we obtain for the boundary conditions $v|_{\Gamma_b \cup \Gamma_l} = 0,\ v|_{\Gamma_p} = \left.\dfrac{\partial v}{\partial x_1}\right|_{\Gamma_p} =$ 0 inequality (1) with

$$\|v\|^2_{+,*} = \int_G \left(|v|^2 + \left| \frac{\partial v}{\partial x_1} \right|^2 \right) dx. \tag{5}$$

Passing to the adjoint boundary conditions, we obtain the theorem:

Theorem 1. *Suppose that on Γ there are no pieces of characteristics of the differential expression $\mathcal{L}$ under consideration. The boundary-value problem $\mathcal{L}[u] = f,\ u|_{\Gamma_b \cup \Gamma_l} = 0,\ u|_{\Gamma_p}$ removed, has a weak solution for every $f \in H^{-,*}$, where $H^{-,*}$ is constructed from the positive norm (5).*

5°. Consider an equation of mixed type generalizing Chaplygin' s equation. Namely, suppose

$$\mathcal{L}[u] = \sum_{j,k=1}^{2} D_j(a_{jk}(x)D_{ku}) + \sum_{j=1}^{2} a_j(x)D_{ju} + a(x)u$$

is given in a two-dimensional domain G, bounded by a piecewise-smooth curve; G intersects the axis Ox_1. Assume that the coefficients of $\mathcal{L}$ are sufficiently smooth and such that, in the domain $G_e = G \cap \{x_2 > 0\}$, $\sum a_{jk}(x)\xi_j\xi_k \geq \varepsilon|\xi_2|^2$ $(\varepsilon > 0)$ and $a(x)$ is sufficiently negative, while in the domain $G_r = G \cap \{x_2 < 0\}$ they pass continuously into the coefficients of the expression $k(x_2)D_1^2 + D_2^2$. Here $k(x_2)$ is continuous on $[-h, 0]$ and continuously differentiable on $[-h, 0)$, with $k(x_2) < 0$, $k'(x_2) > 0$ on $[-h, 0)$, $\lim_{x_2 \to 0} k(x_2)/k'(x_2) = 0$, and $(k/k')'$ is summable on $[-h, 0]$ (h denotes a number so large that G lies in the half-plane $x_2 \geq -h$). In addition, suppose that F. I. Frankl' s condition is satisfied: $2(k/k')' + 1 \geq \delta > 0$ $(x_2 \in [-h, 0])$. The domain G for $x_2 > 0$ is bounded by an arbitrary curve Γ_b, which at the points A_l and A_p approaches the axis Ox_1. For $x_2 < 0$ the boundary consists of two arcs γ_l and γ_p, issuing from A_l and A_p, respectively, and having equations $x_2 = \alpha_l(x_1)$, $x_2 = \alpha_p(x_1)$, where $\alpha_l \geq 0$, $\alpha_l' < (-k)^{-1/2}$; $\alpha_p \leq 0$, $|\alpha_p'| < (-k)^{-1/2}$. The arcs are closed by two pieces Γ_l and Γ_p of characteristics. These arcs may reduce to points; then we obtain the usual domain for an equation of Tricomi type. Developing the method of item 3°, one can prove the theorem:

Theorem 2. *Let $\mathcal{L}$ and G be of the indicated type. The boundary conditions*

$$u|_{\Gamma_b \cup \Gamma_l \cup \gamma_l \cup \gamma_p} = 0, \quad u|_{\Gamma_p}$$

removed are correct; moreover, as the positive norm in inequality (1) we take the norm

$$\|v\|_{+,*}^2 = \int_G |v|^2\,dx + \int_{G_e} \sum_{j,k=1}^{2} a_{jk}(x) D_{kvD} j\overline{v}\,dx + \int_{G_r} \left|\sqrt{-k}\,D_1 v + D_2 v\right|^2\,dx,$$

and in inequality (2) the same norm, except that in the last integral the sign + is replaced by −. The boundary-value problem $\mathcal{L}[u] = f$ with the indicated boundary condition has a weak solution for every $f \in H^{-,*}$.

We note that both $\mathcal{L}$ and G may be taken somewhat more general. The facts set forth in this item are closely connected with the works ([7]) and ([8]), § 18.

6°. For the exposition of the next example it is useful to note that the method of obtaining inequalities of type (2) generalizes to certain equations with respect to vector-functions $u(x)$ with values in a Hilbert space. Thus, let H be a Hilbert space with scalar product (f, g); let $\mathfrak{H}$ denote the Hilbert space of vector-functions $u(t)$ $(t \in [0, T])$ with values in H,

$$(u, v)_0 = \int_0^T (u(t), v(t))\,dt$$

being the scalar product in $\mathfrak{H}$. On $u(t)$ we define the differential

the expression $\mathcal{L}[u] = du/dt - A(t)u$, where the derivative is understood in the strong sense, and $A(t)$ is a linear operator in H with domain of definition $\mathfrak{D}(A(t))$. Let $\mathfrak{D}(\mathcal{L})$ denote the set of all such $u(t) \in \mathfrak{H}$ for which $u(t) \in \mathfrak{D}(A(t))$ and $\mathcal{L}[u] \in \mathfrak{H}$. It is not hard to show that if for all t $\operatorname{Re}(A(t)u, u) \geq 0$ ($u \in \mathfrak{D}(A(t))$), then for $u(t) \in \mathfrak{D}(\mathcal{L})$ satisfying the condition $u(T) = 0$, the inequality $\|\mathcal{L}[u]\|_0 \geq c\|u\|_0$ holds. From this one can obtain the following fact, connected with recent investigations of G. E. Shilov ([9]):

Theorem 3. Let A be a self-adjoint operator in H with resolution of the identity $E(\Delta)$; $\mathcal{L} = d/dt - A$, $\mathcal{L}^+ = -d/dt - A$. For functions $u(t) \in \mathfrak{D}(\mathcal{L})$ we consider the boundary conditions (bc) $u(0) \in E([0, +\infty))H$, $u(T) \in E((-\infty, 0))H$; for $v(t) \in \mathfrak{D}(\mathcal{L}^+)$, the adjoint conditions $(\mathrm{bc})^+$ are $v(0) \in E((-\infty, 0))H$, $v(T) \in E([0, +\infty))H$. These boundary conditions are correct in the sense that the inequalities hold

$$\|\mathcal{L}[u]\|_0 \geq c\|u\|_0, \qquad \|\mathcal{L}^+[v]\|_0 \geq c\|v\|_0.$$

From the second inequality follows the existence of a weak solution of the boundary-value problem $\mathcal{L}[u] = f \in \mathfrak{H}$, $u \in$ (bc), i.e. the existence of such a $u \in \mathfrak{H}$ that for every $v \in \mathfrak{D}(\mathcal{L}^+)$, $v \in (\mathrm{bc})^+$, the relation $(u, \mathcal{L}^+[v])_0 = (f, v)_0$ is satisfied.

7°. In Theorem 3 (as in the assertion preceding it), as A one may take a self-adjoint operator generated by a non-elliptic differential expression; this will lead to nonclassical boundary-value problems. To prove the self-adjointness of A, one may use the results of the note ([10]), in which the corresponding facts are given for ultrahyperbolic expressions in a bounded domain (see also ([11]) for an unbounded domain). We shall give only the following example (cf. ([5]), Ch. 7). Consider the differential expression $\mathcal{L} = D_1 + D_2^2 - D_3^2$ for $(x_1, x_2, x_3) \in G = [0, \pi] \times [0, \pi] \times [0, \pi]$. By Γ_l and Γ_r denote the left and right bases of the parallelepiped G, and by Γ_b its lateral surface.

Theorem 4. For the indicated $\mathcal{L}$ and G the following boundary conditions are correct: $u|_{\Gamma_b} = 0$; $u|_{\Gamma_l} \perp \sin \mu_2 x_2 \cdot \sin \mu_3 x_3$, where the integers μ_2 and μ_3 are such that $\mu_2^2 - \mu_3^2 < 0$; analogously $u|_{\Gamma_r} \perp \sin \mu_2 x_2 \cdot \sin \mu_3 x_3$, $\mu_2^2 - \mu_3^2 \geq 0$ ($\perp$ means orthogonality in $L_2([0, \pi] \times [0, \pi])$). In the inequalities (1) and (2), the norm in L_2 is taken as the positive norm. The boundary-value problem $\mathcal{L}[u] = f$ with the indicated boundary conditions has a weak solution for any $f \in L_2$.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
25 XI 1959

CITED LITERATURE

1. Yu. M. Berezanskii, DAN, **126**, No. 6 (1959).
2. P. D. Lax, Comm. Pure and Appl. Math., **8**, No. 4 (1955); Collection of translations: *Mathematics*, **1**, No. 1 (1957).
3. Yu. M. Berezanskii, DAN, **122**, No. 6 (1958).
4. G. I. Kats, DAN, **119**, No. [[unclear: issue number]] (1958).
5. A. A. Dezin, Uspekhi Mat. Nauk, **14**, No. 3 (1959).
6. L. Hörmander, *On the theory of general differential operators in partial derivatives*, IL, 1959.
7. C. S. Morawetz, Comm. Pure and Appl. Math., **11**, No. 3 (1958).
8. K. O. Friedrichs, Comm. Pure and Appl. Math., **11**, No. 3 (1958).
9. G. E. Shilov, Report at the All-Union Conference on Functional Analysis, Baku, 1959.
10. Yu. M. Berezanskii, Ukr. Mat. Zh., **11**, No. 3 (1959).
11. L. P. Nizhnik, DAN, **124**, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.