MATHEMATICS

R. A. ALEKSANDRYAN

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

R. A. ALEKSANDRYAN

ON OPERATORS GENERATED BY SYSTEMS OF DIFFERENTIAL EQUATIONS OF S. L. SOBOLEV TYPE

(Presented by Academician S. L. Sobolev on 18 XI 1959)

1. It is known that the Cauchy problem in the domain of nonanalytic functions for linear systems of differential equations with partial derivatives of S. V. Kovalevskaya type is, generally speaking, not well posed; and in the work of I. G. Petrovsky (1) a subclass of such systems was singled out, called by him hyperbolic and parabolic, for which the Cauchy problem is uniformly well posed. At the same time it turns out that there exists a certain class of systems, not being systems of S. V. Kovalevskaya type, for which the Cauchy problem is nevertheless well posed. A system of differential equations typical in this respect was first systematically studied by S. L. Sobolev in 1945 and was set forth in detail in (2).

In the present work, for a certain class of such systems, a problem is studied which in its formulation resembles the classical mixed problems, but differs from them in essence, since in it the initial conditions refer only to one group of functions, and the boundary conditions to another. Some spectral properties of the operators generated by this problem are also considered.

2. Consider the system of differential equations:

$$\frac{\partial^k V_i}{\partial t^k} = \sum_{j=1}^{n} \left\{ a_{i,j}(x) V_j + b_{i,j}(x) \frac{\partial P}{\partial x_j} \right\} \quad (i = 1, 2, \dots, n); \qquad \sum_{i=1}^{n} \frac{\partial V_i}{\partial x_i} = 0, \quad (1)$$

where the spatial variables $x = (x_1, x_2, \dots, x_n)$ vary in some bounded domain D with sufficiently smooth boundary Γ, and $t > 0$.

It is assumed that $A(x) = \|a_{i,j}(x)\|$, $B(x) = \|b_{i,j}(x)\|$ are real symmetric and smooth in $\bar{D} = D + \Gamma$ matrices, with $B(x)$ positive definite and $A(x)$ orthogonal at every point of the closed domain $\bar{D}$.

Problem (C) for the system (1) consists in finding such functions $P(t, x)$, $V_i(t, x)$ $(i = 1, 2, \dots, n)$, which satisfy system (1) and the conditions:

$$V_i\Big|_{t=0} = V_i^{(0)}(x),$$

$$\frac{\partial V_i}{\partial t}\Big|_{t=0} = V_i^{(1)}(x), \dots, \frac{\partial^{k-1} V_i}{\partial t^{k-1}}\Big|_{t=0} = V_i^{(k-1)}(x) \quad (i = 1, 2, \dots, n); \tag{2}$$

$$P\Big|_{\Gamma} = 0, \qquad t > 0. \tag{3}$$

Since all subsequent arguments and constructions for different k are completely identical, one may assume, for example, that $k = 2$.

Lemma 1. If $\mathbf{V}(t,x)$ and $P(t,x)$ satisfy system (1), then the function $P(t,x)$ satisfies the equation

$$\partial^2 L(P)/\partial t^2 + M(P) = 0, \tag{4}$$

where $L = -\operatorname{div} B(x)\operatorname{grad}$, $M = -\operatorname{div} A(x)B(x)\operatorname{grad}$.

3. Let Ω be the linear manifold of sufficiently smooth solenoidal vectors $\mathbf{V}(x)$, i.e., those satisfying the last equation of system (1). In the linear manifold Ω we define the scalar product by the formula

$$\big(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}\big) = \int_D B^{-1}(x)\mathbf{V}^{(1)}(x) \cdot \mathbf{V}^{(2)}(x)\, dx, \tag{5}$$

where $B^{-1}(x) = \|b_{i,j}^{-1}(x)\|$ is the matrix inverse to $B(x)$; therefore the norm generated by the scalar product (5) is equivalent to the norm of the space $\mathbf{L}(D)$. Let us close the linear manifold Ω in the scalar product (5), and denote the complete Hilbert space thus obtained by H. The structure of the space H is described by the following lemma.

Lemma 2. A vector-function $\mathbf{V}(x)$ belongs to the space H if and only if $\mathbf{V} \in \mathbf{L}_2(D)$ and

$$(\mathbf{V}, \operatorname{grad}\varphi)_0 = \int_D \sum_{i=1}^{n} V_i(x)\frac{\partial \varphi}{\partial x_i}\, dx = 0, \qquad \varphi(x) \in \Phi_0, \tag{6}$$

where Φ_0 is the linear manifold of functions smooth in $\overline{D}$ that vanish on the boundary Γ.

Let us close the linear manifold Φ_0 in the scalar product defined by the formula

$$(u, v)_B = \int_D B(x)\operatorname{grad} u \cdot \operatorname{grad} v\, dx, \tag{7}$$

and let H_B be the complete Hilbert space thus obtained. The space H_B can also be described by a lemma which is an immediate consequence of one of S. L. Sobolev' s embedding theorems (^{3}).

Lemma 3. A function $\varphi(x)$ belongs to H_B if and only if $\varphi(x) \in W_2^{(1)}(D)$ and vanishes on the boundary Γ in the sense of S. L. Sobolev' s embedding theorems.

Let the operator $\mathfrak{A}$ act in Ω according to the formula

$$\mathfrak{A}\mathbf{V} = A(x)\mathbf{V} + B(x)\,\mathrm{grad}\,P(x),$$

where $P(x)$, for a given $\mathbf{V}(x) \in \Omega$, is the solution of the equation

$$L(P) = \mathrm{div}\,A(x)\mathbf{V}, \tag{8}$$

which satisfies the boundary condition (3).

Thus the operator $\mathfrak{A}$ can also be written in the form

$$\mathfrak{A}\mathbf{V} = A(x)\mathbf{V} + B(x)\,\mathrm{grad}\,S\mathbf{V},$$

where S is the operator which assigns to each $\mathbf{V}(x) \in \Omega$ the unique solution $P(x)$ of the boundary-value problem (8), (3), belonging, evidently, to Φ_0.

Lemma 4. There exists a constant $K_0 > 0$ such that, for $\mathbf{V} \in \Omega$,

$$(S\mathbf{V}, S\mathbf{V})_B \leq K_0(\mathbf{V}, \mathbf{V}).$$

Theorem 1. The operator $\mathfrak{A}$ is bounded and self-adjoint in the Hilbert space H.

Consider the following Cauchy problem, in which it is required to find in H a trajectory $\mathbf{V}(t, x)$ satisfying the operator equation

$$d^2\mathbf{V}/dt^2 = \mathfrak{A}\mathbf{V} \tag{9}$$

and the initial conditions (2).

Theorem 2. The solution of the operator Cauchy problem (9), (2) exists for arbitrary initial data from H, is given by the formula

$$\mathbf{V}(t, x) = \sum_{p=0}^{\infty} \left\{ \frac{t^{2p}}{(2p)!}\mathfrak{A}^p\mathbf{V}^{(0)}(x) + \frac{t^{2p+1}}{(2p+1)!}\mathfrak{A}^p\mathbf{V}^{(1)}(x) \right\}$$

and satisfies the inequality

$$\|\mathbf{V}(t, x)\| \leq e^{t\sqrt{\|\mathfrak{A}\|}} \max\left\{\|\mathbf{V}^{(0)}(x)\|;\ \|\mathbf{V}^{(1)}(x)\|/\sqrt{\|\mathfrak{A}\|}\right\}.$$

Corollary. If $\mathbf{V}(t, x)$ is a solution of the Cauchy problem (9), (2), then $\mathbf{V}(t, x)$ and $P(t, x) = S\mathbf{V}(t, x)$ form a solution of problem (C) for the system (1), and the boundary condition (3) is fulfilled in the sense of Lemma 3. The solution of problem (C) is unique and depends continuously on the initial data in the metric of the space H, uniformly in t on every finite interval.

4. Let the operator Q be defined on the linear manifold Φ_0 by the formula $Qu = -L^{-1}Mu$, where L^{-1} is the operator inverse to the elliptic operator L corresponding to zero boundary conditions.

Theorem 3. The operator Q is symmetric and bounded on Φ_0 in the scalar product (7); therefore it may be regarded as bounded and self-adjoint in the space H_B.

Remark. From Theorem 3 follows the correctness of the formulation of the mixed problem in which one is required to find a solution $P(t, x)$ of equation (4), satisfying the boundary condition (3) and the initial conditions $P|_{t=0} = \psi_0(x)$; $\partial P/\partial t|_{t=0} = \psi_1(x)$.

The study of the behavior of solutions both of problem (C) and of the mixed problem just formulated for large t can be reduced to the study of the spectral properties of the operators $\mathfrak{A}$ and Q, respectively.

In the following section some spectral properties of the operator $\mathfrak{A}$, common to the whole class of systems under consideration, are examined.

5. Let $N(\lambda)$ be the set of points of the domain D at which $\det \|a_{i,j}(x) - \lambda\delta_{i,j}\| = 0$. Obviously, $N(\lambda)$ is a closed subset of the domain $\bar{D}$, which, depending on the matrix $A(x)$ and the value of the parameter λ, may be either empty or coincide with $\bar{D}$. Assume, for simplicity, that the rank of the matrix $A(x) - \lambda E$ is everywhere greater than $n - 2$ for all values of λ. The matrix $A(x)$ may be regarded as an operator which assigns to each $\mathbf{V}(x) \in \mathbf{L}_2(D)$ the vector $A(x)\mathbf{V} \in \mathbf{L}_2$.

Lemma 5. The value of the parameter $\lambda = \lambda_0$ is an eigenvalue of the operator-matrix $A(x)$, considered in all of $\mathbf{L}_2(D)$, if and only if $\operatorname{mes} N(\lambda_0) > 0$.

Construct the vector-function $\mathbf{v}^{(0)}(x)$ as follows: at each point $x \in N(\lambda_0)$, $A(x)\mathbf{v}^{(0)}(x) = \lambda_0\mathbf{v}^{(0)}(x)$, and at these points $\mathbf{v}^{(0)}(x)$ has length one, while everywhere outside $N(\lambda_0)$ we set $\mathbf{v}^{(0)}(x)$ equal to zero. Suppose that $N(\lambda_0)$ is the sum of at most a countable number of closed subdomains $\overline{D_i^{(0)}}$ with piecewise smooth boundaries $\Gamma_i^{(0)}$ $(i = 1, 2, ...)$. Form the function $\mathbf{V}^{(0)}(x) = \alpha(x)\mathbf{v}^{(0)}$, where $\alpha(x)$ is some scalar function.

Lemma 6. The vector $\mathbf{V}^{(0)}(x)$ is an eigenvector of the operator-matrix $A(x)$, considered in the space H, if and only if $\alpha(x) \not\equiv 0$, satisfies inside each of the domains $D_i^{(0)}$ $(i = 1, 2, ...)$ the equation

$$\sum_{k=1}^{n} v_k^{(0)}(x) \frac{\partial \alpha}{\partial x_k} + \operatorname{div} \mathbf{v}^{(0)}(x) \cdot \alpha = 0 \tag{10}$$

and vanishes on those parts of the boundaries $\Gamma_i^{(0)}$ which belong to D.

Theorem 4. The value of the parameter $\lambda = \lambda_0$ is an eigenvalue of the operator-matrix $A(x)$, considered in the space H,

if and only if the current lines of the vector field $\mathbf{v}^{(0)}(x)$ issuing from parts of the boundaries $\Gamma_i^{(0)}$ $(i = 1, 2, ...)$ that belong to the domain D leave uncovered a subset of positive measure of at least one of the domains $D_i^{(0)}$.

Remark. Under the hypotheses of Theorem 4, the corresponding λ_0-eigenspace is infinite-dimensional.

Let λ_k $(k = 1, 2, ...)$ be all those values of the parameter λ for which we are in the hypotheses of Theorem 4, and let the vector fields $\mathbf{v}^{(k)}(x)$ be constructed for the sets $N(\lambda_k)$ in exactly the same way as the field $\mathbf{v}^{(0)}(x)$ was constructed above for the set $N(\lambda_0)$. Let Ω_A be the linear manifold of vectors $\mathbf{V}(x)$ representable in the form of a finite sum

$$\mathbf{V}(x) = \sum_{(k)} \alpha_k(x) \mathbf{v}^{(k)}(x),$$

where the $\alpha_k(x)$ satisfy, inside each of the domains $D_i^{(k)}$ $(i = 1, 2, ...)$, equation (10) with $\mathbf{v}^{(0)}(x)$ replaced by $\mathbf{v}^{(k)}(x)$, and vanish on the parts $\Gamma_i^{(k)}$ belonging to the domain D. Obviously, $\Omega_A \subset H$.

Lemma 7. The linear manifold Ω_A is invariant with respect to the operator $\mathfrak{A}$, and on it the operator $\mathfrak{A}$ coincides with the operator-matrix $A(x)$.

Let

$$H = H_A \oplus H_G,$$

where H_A is the closure of Ω_A in the metric of H.

Lemma 8. In order that $\lambda = \lambda_0$ be an eigenvalue of the operator $\mathfrak{A}$, considered in the subspace H_G, it is necessary and sufficient that the equation

$$\operatorname{div}(A + \lambda_0 E) B \operatorname{grad} u = 0$$

have a nontrivial solution vanishing on the boundary Γ of the domain D.

Theorem 5. The eigenvalues of the operator $\mathfrak{A}$, considered in the subspace H_G, coincide with the eigenvalues of the operator Q in the subspace H_B.

It turns out that a statement holds which substantially supplements this theorem. Let $\mathbf{V}^{(n)} \in H_G$ be a sequence of smooth and normalized vectors such that

$$\lim_{n\to\infty} \|\mathfrak{A}\mathbf{V}^{(n)} - \lambda_0 \mathbf{V}^{(n)}\| = 0,$$

and let

$$P_n(x) = S\mathbf{V}^{(n)}(x);$$

then

$$\lim_{n\to\infty} \|QP_n - \lambda_0 P_n\|_B = 0,$$

whereas $\|P_n\|_B$ does not tend to zero. Conversely, suppose

$$\lim_{n\to\infty} \|QP_n - \lambda_0 P_n\|_B = 0, \qquad \|P_n\|_B = 1;$$

then it is proved that, defining a sequence of vectors $\mathbf{V}^{(n)}(x)$ by the formula

$$\mathbf{V}^{(n)}(x) = (A + \lambda_0 E)B(x)\,\mathrm{grad}\,P_n - B(x)\,\mathrm{grad}(QP_n - \lambda_0 P_n),$$

we shall have

$$\lim_{n\to\infty} \|\mathfrak{A}\mathbf{V}^{(n)} - \lambda_0 \mathbf{V}^{(n)}\| = 0,$$

i.e. the following theorem holds:

Theorem 5 bis. The points of the spectrum of the operator $\mathfrak{A}$, considered in the subspace H_G, coincide with the points of the spectrum of the operator Q in the subspace H_B.

Thus, the space H is representable in the form of an orthogonal sum of subspaces H_A and H_G invariant with respect to the fundamental operator $\mathfrak{A}$. In the subspace H_A, the operator $\mathfrak{A}$ coincides with the operator-matrix $A(x)$ and can, essentially, be studied by considering a certain system of ordinary differential equations.

The investigation of the spectral properties of the operator $\mathfrak{A}$ in the subspace H_G is essentially equivalent to the study of a homogeneous boundary-value problem for general second-order differential operators with, generally speaking, an indefinite quadratic form.

This note is, in a certain sense, a development of works ([2],[4]–[6]), which required a new methodology.

Computing Center
Academy of Sciences of the Armenian SSR

Received
17 XI 1959

References

1. I. G. Petrovsky, *Bull. Moscow State Univ.*, issue 7 (1938).
2. S. L. Sobolev, *Izv. Acad. Sci. USSR, Ser. Math.*, **18**, 3 (1954).

3. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
4. R. A. Aleksandryan, Dissertation, Moscow State University, 1949.
5. R. A. Aleksandryan, *Dokl. Akad. Nauk SSSR*, **73**, No. 4, 7 (1950).
6. R. A. Aleksandryan, *Izv. Acad. Sci. Armenian SSR, Phys.-Math. Ser.*, **10**, No. 1 (1957).

Note: Figure translations are in progress. See original paper for figures.