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Abstract

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PRIMITIVELY FACTORIZABLE GROUPS

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A subgroup $\mathfrak{A}$ of a group $\mathfrak{G}$ is called a **complement** in $\mathfrak{G}$ if there exists in $\mathfrak{G}$ a subgroup $\mathfrak{B}$ such that $\mathfrak{A}\mathfrak{B} = \mathfrak{G}$ and $\mathfrak{A} \cap \mathfrak{B} = 1$.

Arbitrary groups in which all subgroups are complemented, as was shown by N. V. Baeva (Chernikova) in papers ([1,2]) (following her, we shall call them **completely factorizable**), are exhausted by certain periodic subgroups of complete direct products of finite groups whose orders are not divisible by the square of a prime number, and even by complete direct products of specially chosen groups of this kind (called in the present paper completely primitive; see the definitions). Completely factorizable groups were studied by N. V. Chernikova to completion.

Groups with various kinds of systems of complemented subgroups under various additional conditions were studied by S. N. Chernikov in paper ([3]), where for the first time the general question was raised of studying groups with prescribed systems of complemented subgroups. In the present paper such a system is (essentially) the set of all subgroups of prime orders belonging to a given set π of prime numbers; it is considered in connection with the question, proposed to the author by S. N. Chernikov, on the structure of a group in which every p-subgroup is complemented for every p from π.

1. **Definitions.** Let π be some set of prime numbers. Sometimes, as π, we shall use the set π_n of the first n primes of the natural series.

We shall call a group $\mathfrak{G}$ **primitively** π**-factorizable** if all cyclic subgroups of prime orders contained in the set π are complemented in it; in particular, **primitively factorizable** if the set π contains all prime divisors of the orders of elements of the group $\mathfrak{G}$. If the set π consists of a single prime p, then the group $\mathfrak{G}$ will be called **primitively** p**-factorizable**.

By adding to the terms defined above the term **locally**, we shall thereby indicate that every subgroup generated by a finite number of elements is factorizable in the sense of the chosen term.

We shall call a group $\mathfrak{G}$ **locally completely factorizable** if all its subgroups generated by a finite number of elements are completely factorizable.

These definitions, except for the last, plainly do not impose restrictions of periodicity on the classes of groups introduced by them.

A group factorizable in one or another sense is locally factorizable in the same sense. This follows from the following proposition contained in paper ([3]): if a subgroup $\mathfrak{A}$ of a group $\mathfrak{G}$ contains a subgroup $\mathfrak{B}$, a complement of which in the group $\mathfrak{G}$ is the group $\mathfrak{R}$, then the intersection of the groups $\mathfrak{A}$ and $\mathfrak{R}$ complements the subgroup $\mathfrak{B}$ in the group $\mathfrak{A}$.

Let $\mathfrak{P}$ be a group of prime order p. We shall call a completely factorizable subgroup $\mathfrak{A}$ of the holomorph of the group $\mathfrak{P}$ **completely p-primitive** if it contains the group $\mathfrak{P}$. If there is no need to indicate the order of the group $\mathfrak{P}$, then we shall call the group $\mathfrak{A}$ simply **completely primitive**.

The choice of this term is justified by the following proposition (which follows from Theorem 98 of the book [4] by virtue of Theorem 1 of [2]): in order that a finite completely factorizable group $\mathfrak{G}$ be representable as a transitive primitive permutation group, it is necessary and sufficient that it be completely primitive.

We shall call a group $\mathfrak{G}$ **completely p-approximable** if, for every element $P \in \mathfrak{G}$ of prime order p, there exists a normal divisor $\mathfrak{N}_p$ not containing this element such that the factor group $\mathfrak{G}/\mathfrak{N}_p$ is completely p-primitive. If the group $\mathfrak{G}$ is completely p-approximable for every p in the set π, then we shall call the group $\mathfrak{G}$ **completely π-approximable**.

In the case when, for every nonidentity element $P \in \mathfrak{G}$, there exists a normal divisor $\mathfrak{N}_p$ such that the element P is not contained in the group $\mathfrak{N}_p$ and the group $\mathfrak{G}/\mathfrak{N}_p$ is completely primitive, we shall call the group $\mathfrak{G}$ **completely approximable**.

From Theorems 1 and 2 of the present paper, by virtue of Theorem 3 of [2], it follows that, in the definitions of completely p-approximable and completely approximable groups, the requirement that the group $\mathfrak{G}/\mathfrak{N}_p$ be completely primitive may be replaced by the weaker requirement of its complete factorizability.

It is clear that every subgroup of a completely approximable (completely π-approximable) group is completely approximable (completely π-approximable).

A factor group of a nonperiodic completely approximable (completely π-approximable) group need not be completely approximable (completely π-approximable). This can be seen from the example of the infinite cyclic group. A homomorphic image of a periodic completely approximable group is completely approximable (the same is true for completely π-approximable groups).

2. Theorem 1. *An infinite group $\mathfrak{G}$ is completely approximable if and only if it is isomorphic to a subgroup of a complete direct product of completely primitive groups.*

A periodic completely approximable group $\mathfrak{G}$ is embeddable in a direct product of completely primitive groups if and only if it is locally normal.

In particular, a finite group $\mathfrak{G}$ is completely approximable if and only if it is isomorphic to a subgroup of a direct product of a finite number of completely primitive groups.

Theorem 2. *A group $\mathfrak{G}$ is completely π-approximable if and only if it is an extension of a group containing no elements of order $p \in \pi$ by a subgroup of a complete direct product of completely p-primitive groups for $p \in \pi$.*

3. Theorem 3. *An infinite group $\mathfrak{G}$ is primitively p-factorizable if and only if it is an extension of a group containing no elements of order p by a subgroup of a complete direct product of groups isomorphic to the permutation group on p symbols.*

In the case of a finite group $\mathfrak{G}$, one should here take a finite direct product.

Corollary. *If an infinite group $\mathfrak{G}$ is primitively π-factorizable, then it is an extension of a group containing no elements of order $p \in \pi$ by a subgroup of a complete direct product of permutation groups on p symbols, $p \in \pi$.*

In the case of a finite group $\mathfrak{G}$, one should here take a finite direct product.

Remark. Whether an extension of a group $\mathfrak{N}$, containing no elements of order $p \in \pi$, by a subgroup $\mathfrak{A}$ of the complete direct product of groups isomorphic to the permutation group on p symbols, $p \in \pi$, is primitively π-factorizable depends not only on the groups $\mathfrak{N}$ and $\mathfrak{A}$, but also on the way in which the group $\mathfrak{N}$ is extended by the group $\mathfrak{A}$.

In fact, below an example is given of a primitively π_5-factorizable group $\mathfrak{G}$, which is a central extension of a torsion-free group $\mathfrak{Z}$ by means of a subgroup $\mathfrak{P}$ of order 52, holomorphic to a group of order 13. The direct product of the groups $\mathfrak{P}$ and $\mathfrak{Z}$ is not primitively π_5-factorizable. The non-reversibility of the consequence of Theorem 3 is connected with the fact that, in the case of nonperiodic groups, primitive π-factorizability is not carried over to factor groups (see the example).

4. **Theorem 4.** *A periodic group $\mathfrak{G}$ is primitively π_n-factorizable if and only if it is fully π_n-approximable.*

Remark. In the formulation of Theorem 4, the condition of periodicity is essential, as the following example shows.

Example. Let the group $\mathfrak{G}$ be generated by elements A and B satisfying the relations: 1) $A^{13} = 1$; 2) $B^{-1}AB = A^5$.

The group $\mathfrak{G}$ is primitively π_5-factorizable, but is not fully π_5-approximable. The factor group $\mathfrak{G}/\mathfrak{Z}$ of the group $\mathfrak{G}$ by its center $\mathfrak{Z} = \{B^4\}$ is isomorphic to a subgroup $\mathfrak{P}$ of order 52, holomorphic to a group of order 13, and therefore is not primitively π_5-factorizable (since it contains a cyclic subgroup of order 2^2).

5. Let us consider the complete direct product of groups isomorphic to the permutation group on 3 letters. From Theorems 2 and 4 it follows that the group $\mathfrak{G}$ is primitively factorizable. But, as N. V. Chernikova showed ([2]), the group $\mathfrak{G}$ is not fully factorizable. Hence it follows that the class of

periodic primitively factorizable groups is broader than the class of fully factorizable groups.

Since such a group $\mathfrak{G}$ is, obviously, locally fully factorizable, the property of full factorizability is not local. The following proposition shows that the property of primitive π-factorizability, in contrast to the property of full factorizability, is local.

Theorem 5. *A group $\mathfrak{G}$ is primitively π-factorizable if and only if it is locally primitively π-factorizable.*

6. **Theorem 6.** *The following classes of groups coincide:*

1) *the class of periodic primitively factorizable groups;*

2) *the class of periodic fully approximable groups;*

3) *the class of periodic subgroups of complete direct products of fully primitive groups;*

4) *the class of locally fully factorizable groups;*

5) *the class of periodic groups possessing an invariant system with cyclic factors and Sylow elementary abelian p-subgroups.*

Remark. By virtue of S. N. Chernikov' s theorem on the local finiteness of a periodic solvable group ([5]), from the coincidence of classes 1 and 3 there follows the local finiteness of periodic primitively factorizable groups.

In view of S. N. Chernikov' s theorem ([3]) on the full factorizability of a locally finite group with complemented abelian subgroups, it follows from this that there is a positive solution of the question posed by S. N. Chernikov in ([3]) (and already solved by M. I. Kargapolov ([6])) concerning the equivalence of the condition of full factorizability of a group and the condition that all its abelian subgroups be complemented.

We note that Theorem 6 generalizes all the results of Ph. Hall from the paper ([7]).

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Note: Figure translations are in progress. See original paper for figures.