MATHEMATICS

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Abstract

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FOURIER TRANSFORMS OF RAPIDLY DECREASING FUNCTIONS ON COMPLEX SEMISIMPLE GROUPS

1. The spaces $\mathcal{D}_\chi$. It is more convenient to define irreducible representations of complex semisimple Lie groups not in Banach spaces, but in the linear topological spaces $\mathcal{D}_\chi$, which we shall now describe. Let G be a complex semisimple connected Lie group, D its Cartan subgroup, and let Z and Z^- be the subgroups generated respectively by the positive and negative root vectors. Introduce the space $H = G/Z$ of right cosets of G modulo Z ("the basic affine space" ; by $\tilde{g}$ we shall denote the coset in H containing $g \in G$. The elements $g_0 \in G$ define motions in H, under which the coset $\tilde{g}$ passes into $\widetilde{gg_0}$; on H there exists a measure $d\tilde{g}$, invariant with respect to these motions. On the other hand, every element $\delta \in D$ defines in H a "similarity transformation" $\tilde{g} \to \widetilde{\delta g}$, commuting with the motions (it is not hard to see that the coset $\widetilde{\delta g}$ does not depend on the choice of the representative in $\tilde{g}$). Put $d(\widetilde{\delta g})/d\tilde{g} = \beta(\delta)$.

We shall call the space $\mathcal{D}_\chi$, where $\chi(\delta)$ is a complex character on D (i.e. a homomorphism of the group D into the group of complex numbers under multiplication), the space of all continuous infinitely differentiable functions $\varphi(\tilde{g})$ on H of degree of homogeneity $\chi(\delta)\beta^{-1/2}(\delta)$, i.e. such that

$$\varphi(\widetilde{\delta g}) = \varphi(\tilde{g})\chi(\delta)\beta^{-1/2}(\delta).$$

The topology in $\mathcal{D}_\chi$ is defined in the natural way.

The space $\mathcal{D}_\chi$ can be realized in another way. Introduce the "basic projective space" $Z' = G/ZD$, whose points are the "lines" $\widetilde{\delta g}$ in H, where δ runs through D. Decompose Z' into N submanifolds: the set of lines δz, $z \in Z$, identical with Z, and $N-1$ sets of lines $\delta s_i z$, where s_i is an element of the Weyl group, $s_i \neq 1$ (N is the order of the Weyl group). The last $N-1$ sets have lower dimension and may be interpreted as sets of "infinitely distant points" in Z'. We define $\mathcal{D}_\chi$ as a space of functions on Z, replacing the functions $\varphi(\tilde{g})$ on H by functions $f(z) = \varphi(\tilde{z})$. The functions $f(z)$ from $\mathcal{D}_\chi$ are infinitely differentiable and, as z tends to infinity, have $N-1$ prescribed asymptotics (according to the number of types of infinitely distant points). In particular, if as $z \to \infty$ one can choose $\delta = \delta(z) \in D$ such that $\lim \delta z = \tilde{g}_0$, then for large z we have

$$f(z) \sim C\chi^{-1}(\delta)\beta^{1/2}(\delta).$$

2. Representations in $\mathcal{D}_\chi$. With each space $\mathcal{D}_\chi$ we associate a representation of the group G:

$$T^{\chi}_{g_0}\varphi(\tilde{g}) = \varphi(\widetilde{g g_0}), \qquad \tilde{g} \in H.$$

If we define $\mathcal{D}_\chi$ not as a space of functions on H, but as a space of functions $f(z)$ on Z, then the formula for the operator takes the form

$$T^{\chi}_{g_0} f(z) = f(\overline{zg})\chi(\delta)\beta^{-1/2}(\delta),$$

where $\overline{zg} = z' \in Z$ and $\delta \in D$ are determined from the relation $zg = \zeta\delta z'$, $\zeta \in Z$.

The representations of the group G in $\mathcal{D}_\chi$ are irreducible for all χ, with the exception of a set of special characters of lower dimension;

for these special characters χ, the representations in $\mathcal{D}_\chi$ turn out to be reducible. We shall seek an operator B carrying $\mathcal{D}_{\chi_1}$ into $\mathcal{D}_{\chi_2}$ and intertwining the representation operators: $BT^{\chi_1}_g = T^{\chi_2}_g B$ for any $g \in G$. It turns out that such an operator exists and has an inverse if and only if χ_1 and χ_2 are related by the relation $\chi_2(\delta) = \chi_1(\delta^s)$, where $\delta \to \delta^s$ is a certain automorphism from the Weyl group. In this case the representations in $\mathcal{D}_{\chi_1}$ and $\mathcal{D}_{\chi_2}$ are equivalent. Moreover, there are special cases when an intertwining operator exists but has no inverse. In these cases (they are discussed at the end of the paper) the representations in $\mathcal{D}_{\chi_1}$ and $\mathcal{D}_{\chi_2}$ will be called semi-equivalent (see also ([2])).

We can now speak of the "space" of pairwise inequivalent representations of the group G in the spaces $\mathcal{D}_\chi$. The points at which these representations turn out to be semireducible are special points of the "space of representations." The "universal covering" for the space of representations is the analytic manifold X of all complex characters $\chi(\delta)$ on D. The Weyl group S is the analogue of the fundamental group.

3. The basic group ring Γ and the Fourier transforms of functions from Γ. We shall call a function $x(g)$ on the group G rapidly decreasing on G if $|x(g)| = o(\|l_g\|)$ for any finite-dimensional representation l_g of the group G in a unitary space.* By derivatives of the function $x(g)$ we shall mean expressions $P(X_i)x(g)$, where $P(X_i)$ is an arbitrary polynomial in the Lie operators of left and right translations on G. Let Γ be the collection of all such infinitely differentiable functions $x(g)$ on G such that $P(X_i)x(g)$ is a rapidly decreasing function for any $P(X_i)$. We define multiplication in Γ as convolution and introduce in

Γ the topology in the natural way. The resulting ring will be called the **basic group ring** of the group G.

Let $x(g) \in \Gamma$. Then for any character $\chi = \chi(\delta)$ the operator

$$T_x^\chi = \int x(g) T_g^\chi \, dg$$

is an operator in $\mathcal{D}_\chi$ with kernel $K(z_1, z_2; \chi)$, where $z_1, z_2 \in Z$. We shall call this kernel the **Fourier transform** of the function $x(g)$. The correspondence $x(g) \to K(z_1, z_2; \chi)$ carries the ring Γ into the ring of kernels with the usual multiplication operation for kernels. The problem is to describe the kernels $K(z_1, z_2; \chi)$. In paragraphs 4-6 necessary and sufficient conditions are formulated on a function $K(z_1, z_2; \chi)$ under which it is the Fourier transform of some function from Γ.**

4. Membership of $K(z_1, z_2; \chi)$ in the space $\mathcal{D}_\chi \otimes \mathcal{D}_{\chi^{-1}}$. The kernel of an operator in the space $\mathcal{D}_\chi$, including the operator T_x^χ, is an element of the tensor product $\mathcal{D}_\chi \otimes \mathcal{D}'_\chi$ of the space $\mathcal{D}_\chi$ and the space $\mathcal{D}'_\chi$ conjugate to it. We now note that the space $\mathcal{D}_{\chi^{-1}}$, where $\chi^{-1}(\delta) = \chi(\delta^{-1})$, is embedded in a natural way in $\mathcal{D}'_\chi$, since any function from $\mathcal{D}_{\chi^{-1}}$ defines a linear functional in $\mathcal{D}_\chi$. It turns out that the Fourier transform $K(z_1, z_2; \chi)$ of a function $x(g) \in \Gamma$ belongs to the subspace $\mathcal{D}_\chi \otimes \mathcal{D}_{\chi^{-1}}$ of the space $\mathcal{D}_\chi \otimes \mathcal{D}'_\chi$. Thus, $K(z_1, z_2; \chi)$ is an infinitely differentiable function of z_1 and z_2 and satisfies, for large z_1, z_2, $N^2 - 1$ asymptotic relations, where N is the order of the Weyl group.

5. Growth conditions for the function $K(z_1, z_2; \chi)$. The function $K(z_1, z_2; \chi)$, as well as its derivatives of any order with respect to the parameters of the elements $z_1, z_2 \in Z$

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* If the group G is realized as a group of matrices, then this requirement is equivalent to the following: $|x(g)| = o(\|g\|^{-n})$ for any $n > 0$, where $\|g\|^2 = \mathrm{Sp}(gg^*)$.

** For the case when G is the group of complex matrices of the 2nd order with determinant 1, this problem was solved in ([1]), and then by another method in ([5]). Fourier transforms of functions on the group of real matrices of the 2nd order with determinant 1 were obtained by other methods in ([3,4]).

are entire functions of χ. Their description is more convenient in terms of the "Mellin transforms" $\varphi(z_1, z_2; \delta)$, related to $K(z_1, z_2; \chi)$ by the relation

$$K(z_1, z_2; \chi) = \int \varphi(z_1, z_2; \delta) \chi(\delta) \, d\delta,$$

where $\delta \in D$, and the integral is taken with respect to the invariant measure.

Namely, for any $n > 0$ the function $\varphi(z_1, z_2; \delta)$ satisfies the condition

$$|\varphi(z_1, z_2; \delta)| = o(\|\delta\|^{-n})$$

uniformly in every bounded domain with respect to z_1 and z_2.

The functions

$$\varphi_{g_1,g_2}(z_1, z_2; \delta) = \varphi(z_1 g_1, z_2 g_2, \delta_1^{-1}\delta\delta_2)\, \beta^{-1/2}(\delta_1)\, \beta^{-1/2}(\delta_2),$$

where the elements $z_1 g_1 = z_1'$ and $z_2 g_2 = z_2'$ from Z and $\delta_1, \delta_2 \in D$ are determined by the equalities

$$z_1 g_1 = \zeta_1 \delta_1 z_1', \qquad z_2 g_2 = \zeta_2 \delta_2 z_2',$$

$\zeta_1, \zeta_2 \in Z$, satisfy the same condition for any $g_1, g_2 \in G$. Moreover, it suffices to require that g_1, g_2 run over one representative from each double coset of the group G modulo the subgroup $K = ZD$ (this gives N^2 conditions, where N is the order of the Weyl group).

6. **Relations between the kernels $K(z_1, z_2; \chi_1)$ and $K(z_1, z_2; \chi_2)$.** For some characters χ_1, χ_2 there exists an operator B, mapping $\mathfrak{D}_{\chi_1}$ into $\mathfrak{D}_{\chi_2}$, such that

$$BK(z_1, z_2; \chi_1) = K(z_1, z_2; \chi_2)B. \tag{1}$$

To describe these relations, introduce the decomposition of the manifold Z into submanifolds. Let $N(D)$ be the normalizer of the subgroup D in G. The factor group $N(D)/D$ (the "Weyl group") consists of a finite number of elements; let $s_1, \dots, s_N$ be representatives of the conjugacy classes of $N(D)$ modulo D. To each conjugacy class of $N(D)$ modulo D we assign the manifold Z_i of elements $z_i \in Z$ representable in the form

$$z_i = \zeta_1 s_i \delta \zeta_2,$$

where s_i is a representative of the conjugacy class, $\zeta_1, \zeta_2 \in Z$, and $\delta \in D$. These manifolds Z_i are pairwise disjoint, and their union is all of Z.

6.1. **Relations of the general type.** If the characters χ_1 and χ_2 are related by

$$\chi_1(\delta^{s_i}) = \chi_2(\delta),$$

where

$$\delta^{s_i} = s_i \delta s_i^{-1},$$

then the kernels $K(z_1, z_2; \chi_1)$ and $K(z_1, z_2; \chi_2)$ are related by (1). The operator B is given by the kernel $B(z_1, z_2) = B(z_2 z_1^{-1})$; the function $B(z)$ is concentrated on Z_i and is given on Z_i by the formula

$$B(z_i) = \chi_2^{-1}(\delta_i)\beta^{1/2}(\delta_i), \tag{2}$$

where $\delta_i \in D$ is determined from the relation

$$z_i = \zeta_1 s_i \delta_i \zeta_2.$$

Formula (2) defines B on the whole space of characters X, except at special points. At special points $B(z_i)$ is defined as the residue of the generalized function of z_i

$$\chi^{-1}(\delta_i)\beta^{1/2}(\delta_i)$$

at the value of the parameter $\chi = \chi_2$. In this case $B(z_i)$ is a generalized function concentrated on one of the manifolds bordering on Z_i.

The operator B has an inverse operator. Thus, relations of the general type express the equivalence of representations in the spaces $\mathfrak{D}_{\chi_1}$ and $\mathfrak{D}_{\chi_2}$.

6.2. **Special relations.** At special points of the character space X, further special relations between the kernels $K(z_1, z_2; \chi_1)$ and $K(z_1, z_2; \chi_2)$ are possible. Consider the function

$$a(z_i, \chi) = \chi(\delta_i)$$

on the manifold Z_i, where χ is a complex character on D, and δ_i is determined from the relation

$$z_i = \zeta_1 s_i \delta_i \zeta_2.$$

We shall say that an element s_j of the Weyl group is subordinate to an element s_i if $a(z_i, \chi)$ has, for some $\chi = \chi_0$, a pole whose residue is a δ-function concentrated on Z_j. To each pair (s_j, s_i), where s_j is subordinate to s_i, there corresponds a set of special relations between the kernels. Decompose each complex character χ into the product

$$\chi(\delta) = \chi'(\delta)\chi''(\delta)$$

of its analytic and antianalytic components. Special relations

relate the kernels $K(z_1, z_2; \chi_1)$ and $K(z_1, z_2, \chi_2)$, where $\chi_1'(\delta^{s_i}) = \chi_2'(\delta)$ and $\chi_1''(\delta^{s_i}) = \chi_2''(\delta)$ (or $\chi_1'(\delta^{s_i}) = \chi_2'(\delta)$ and $\chi_1''(\delta^{s_j}) = \chi_2''(\delta)$). In addition, the characters χ_1 and χ_2 satisfy one further requirement, which will be indicated below. Let us describe the corresponding operator B. Consider on Z_i the function $\alpha_i(z_i) = \overline{\chi_2}^{-1}(\delta_i)\beta^{1/2}(\delta_i)$, where δ_i is determined from the relation $z_i = \zeta_1 s_i \delta_i \zeta_2$. Similarly define the function $\alpha_j(z_j)$. We decompose the functions $\alpha_i(z_i)$ and $\alpha_j(z_j)$ into products of analytic and anti-analytic components $\alpha_i(z_i) = \alpha_i'(z_i)\overline{\alpha_i''(z_i)}$, $\alpha_j(z_j) = \alpha_j'(z_j)\overline{\alpha_j''(z_j)}$, and then extend the functions $\alpha_j'(z_j)$, $\alpha_j''(z_j)$ to analytic functions $\alpha_j'(z_i)$, $\alpha_j''(z_i)$ on Z_i. By assumption, there exists such a character $\chi_0(\delta) = \chi_0'(\delta)\overline{\chi_0''(\delta)}$ that the residue of $\chi(\delta_i)$ for $\chi = \chi_0$, where δ_i is determined from the relation $z_i = \zeta_1 s_i \delta_i \zeta_2$, is a δ-function concentrated on Z_j. The required operator B is given by the kernel $B(z_1, z_2) = B(z_2 z_1^{-1})$, where $B(z)$ is the generalized function on Z_i defined by the formula

$$B(z_i) = \operatorname{Res} \alpha_j'(z_i)\overline{\alpha_i''(z_i)}\,\overline{\chi_0''}^{-1}(\delta_i)\chi(\delta_i) \quad \text{for } \chi = \chi_0 \tag{3}$$

(or, respectively, by the formula

$$B(z) = \text{Res}\ \overline{\alpha_j''(z_i)}\alpha_i'(z_i)\chi_0'^{-1}(\delta_i)\chi(\delta_i) \quad \text{for } \chi = \chi_0).$$

The additional condition on the character χ_2 (and thereby also on the character χ_1, by virtue of the relations between χ_1 and χ_2) consists in the fact that $B(z_i)$ must be a generalized function concentrated on Z_j (or on a submanifold subordinate to Z_j).

The special relations between kernels are connected with the fact that representations in the spaces $\mathfrak{D}_\chi$, where χ is a special character, are semireducible. In this case, in the various spaces $\mathfrak{D}_{\chi_1}$ and $\mathfrak{D}_{\chi_2}$ there may occur invariant subspaces or quotient spaces in which the representations are equivalent. In particular, the special relations completely describe all invariant subspaces in the semireducible spaces $\mathfrak{D}_\chi$.

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Note: Figure translations are in progress. See original paper for figures.