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Abstract

Full Text

A BASIS OF ROOT VECTORS OF A DISSIPATIVE OPERATOR*

A. S. MARKUS

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Let A be a linear operator acting in a separable Hilbert space \mathfrak{H} . A vector x is called a **root vector** of the operator A , corresponding to the number λ , if there exists a natural number n such that $(A - \lambda I)^n x = 0$. The set \mathfrak{S}_λ of all root vectors of the operator A corresponding to the number λ is called the **root linear manifold** (or, if it is closed, the **root subspace**) of the operator A , corresponding to the number λ . The operator A is called **dissipative** if $\text{Im}(Ax, x) \geq 0$ for all $x \in \mathfrak{D}_A$.

In the paper of B. R. Mukminov ⁽¹⁾, sufficient conditions were indicated for the system of eigenvectors of a dissipative operator to be a Riesz basis ⁽²⁾ of its closed linear span. I. M. Glazman ⁽³⁾ obtained an elementary proof of this result (not using M. S. Livshits' triangular model of dissipative operators) under less restrictive conditions. In the present note we indicate certain conditions under which a dissipative operator has a basis of root vectors.

1. In what follows we shall need the notion of a basis of subspaces ⁽⁴⁾. A sequence $\{\mathfrak{R}_k\}_1^\infty$ of nonzero subspaces is called a **basis** of the space \mathfrak{H} if every vector $x \in \mathfrak{H}$ is represented uniquely in the form of a series

$$x = \sum_{k=1}^{\infty} x_k,$$

where $x_k \in \mathfrak{R}_k$. If the subspaces \mathfrak{R}_k are pairwise orthogonal, the basis is called **orthogonal**. If for the sequence of subspaces $\{\mathfrak{R}_k\}_1^\infty$ there exists a linear bounded continuously invertible operator A and an orthogonal basis of subspaces $\{\mathfrak{M}_k\}_1^\infty$ such that $A\mathfrak{M}_k = \mathfrak{R}_k$, then, obviously, $\{\mathfrak{R}_k\}_1^\infty$ is a basis. We shall call it a **Riesz basis**. The notion of a Riesz basis was introduced and studied for vector bases by N. K. Bari ⁽²⁾. Riesz bases of subspaces were considered by M. K. Fage ⁽⁴⁾ (he called them straightenable). With the aid of the results of the paper of I. M. Gelfand ⁽⁵⁾ it is not difficult to establish that a basis of subspaces is a Riesz basis if and only if every permutation of it is also a basis.

We shall call a sequence of nonzero subspaces $\{\mathfrak{R}_k\}_1^\infty$ **ω -linearly independent** if from the equality

$$\sum_{k=1}^{\infty} x_k = 0 \quad (x_k \in \mathfrak{N}_k)$$

it follows that $x_k = 0$ ($k = 1, 2, \dots$). A sequence of nonzero sub-

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We shall call a sequence of subspaces $\{\mathfrak{M}_k\}_1^{\infty}$ **separated** if, for every natural k , the minimal angle between the subspace \mathfrak{M}_k and the closed linear span of all the remaining subspaces \mathfrak{M}_i ($i \neq k$) is positive. Recall that the **minimal angle** $\varphi(\mathfrak{M}, \mathfrak{N})$ ($0 \leq \varphi \leq \pi/2$) between subspaces \mathfrak{M} and \mathfrak{N} is defined by the equality⁽⁶⁾

$$\cos \varphi(\mathfrak{M}, \mathfrak{N}) = \sup_{\substack{x \in \mathfrak{M}, y \in \mathfrak{N} \\ |x|=|y|=1}} |(x, y)|.$$

We note that every separated sequence of subspaces is ω -linearly independent and that every basis is separated.

We shall call two sequences of subspaces $\{\mathfrak{M}_k\}_1^{\infty}$ and $\{\mathfrak{N}_k\}_1^{\infty}$ **quadratically close** if

$$\sum_{k=1}^{\infty} |P_k - Q_k|^2 < \infty,$$

where P_k (Q_k) is the orthogonal projector onto the subspace \mathfrak{M}_k (\mathfrak{N}_k).

The following theorem is an analogue of the well-known theorem of N. K. Bari⁽⁷⁾ on the stability of orthonormalized bases.

Theorem 1. *A separated sequence of subspaces $\{\mathfrak{M}_k\}_1^{\infty}$, quadratically close to some orthogonal basis $\{\mathfrak{N}_k\}_1^{\infty}$, is a Riesz basis of its closed linear span.*

We note that, unlike the vector bases considered by N. K. Bari, the sequence $\{\mathfrak{M}_k\}_1^{\infty}$, generally speaking, is not complete.*

A basis of subspaces quadratically close to some orthogonal basis will be called a **Bari basis**. The concept of a Bari basis was introduced and studied for vector bases by M. G. Krein⁽⁸⁾.

Theorem 2. *If a complete separated sequence of subspaces $\{\mathfrak{M}_k\}_1^{\infty}$ satisfies the condition*

$$\sum_{\substack{j,k=1 \\ j \neq k}}^{\infty} \cos^2 \varphi(\mathfrak{M}_j, \mathfrak{M}_k) < \infty, \quad (1)$$

then it is a Bari basis.

This theorem is a partial generalization of the criterion for a Bari basis of vectors obtained by M. G. Krein (8). Unlike vector bases, condition (1) is not necessary, even in the case when the subspaces \mathfrak{M}_k ($k = 1, 2, \dots$) are finite-dimensional.**

We note that if the subspaces \mathfrak{M}_k ($k = 1, 2, \dots$) appearing in Theorems 1 and 2 are assumed finite-dimensional, then the separability condition may be replaced by the condition of ω -linear independence.

If the sequence of subspaces $\{\mathfrak{M}_k\}_1^\infty$ is a Riesz basis, then, obviously, the union of orthonormalized bases of the subspaces \mathfrak{M}_k ($k = 1, 2, \dots$) is a vector Riesz basis. If, however, the sequence $\{\mathfrak{M}_k\}_1^\infty$ is a Bari basis, then, as is easy to see, the sequence of vectors composed of orthonormalized bases of the subspaces \mathfrak{M}_k ($k = 1, 2, \dots$), while being a Riesz basis, is not, generally speaking, a Bari basis. We shall indicate one sufficient condition under which the indicated sequence of vectors will be a Bari basis.

* If all subspaces \mathfrak{N}_k are finite-dimensional and $\dim \mathfrak{N}_k = \dim \mathfrak{M}_k$ ($k = 1, 2, \dots$), then the sequence $\{\mathfrak{M}_k\}_1^\infty$ will be complete.

** Condition (1) will be necessary, for example, if the dimensions of all subspaces \mathfrak{M}_k are bounded by one number.

Theorem 3. If $\{\mathfrak{N}_k\}_1^\infty$ is a complete ω -linearly independent sequence of finite-dimensional subspaces ($n_k = \dim \mathfrak{N}_k < \infty$), and if

$$\sum_{\substack{j,k=1 \\ j \neq k}}^{\infty} \min(n_j, n_k) \cos^2 \varphi(\mathfrak{N}_j, \mathfrak{N}_k) < \infty,$$

then the union of the orthonormal bases of the subspaces \mathfrak{N}_k ($k = 1, 2, \dots$) is a Bari basis.

2. Let λ be an eigenvalue of a linear operator A . By ν_λ we shall denote the multiplicity of the eigenvalue λ , i.e. the dimension of the corresponding root subspace \mathfrak{E}_λ . If there exists a natural number μ such that $(A - \lambda I)^\mu \mathfrak{E}_\lambda = 0$, then the least of these numbers will be denoted by μ_λ ; otherwise we put $\mu_\lambda = \infty$. The number μ_λ will be called the order of the eigenvalue λ . Note that if the operator A is closed, then the order μ_λ is finite if and only if \mathfrak{E}_λ is a subspace.

Lemma. If A is a dissipative operator, and λ and τ are two of its eigenvalues of finite order, then

$$\cos \varphi(\mathfrak{E}_\lambda, \mathfrak{E}_\tau) \leq \frac{2\sqrt{\mu_\lambda \mu_\tau} \operatorname{Im} \lambda \operatorname{Im} \tau}{|\lambda - \bar{\tau}|} \sum_{j=0}^{\mu_\lambda-1} a^j \sum_{k=0}^{\mu_\tau-1} b^k,$$

where

$$a = \frac{2\sqrt{2}(\mu_\lambda - 1) \operatorname{Im} \lambda}{|\lambda - \bar{\tau}|}, \quad b = \frac{2\sqrt{2}(\mu_\tau - 1) \operatorname{Im} \tau}{|\lambda - \bar{\tau}|}.$$

This lemma is a generalization of the estimate obtained by I. M. Glazman [3] for eigenvectors.

Theorem 4. Let A be a closed linear operator possessing a sequence of eigenvalues $\{\lambda_k\}_1^\infty$ ($\lambda_k \neq \lambda_j$ for $k \neq j$). If the following conditions are satisfied:

- 1) $A = H + iT$, where H is self-adjoint, and T is a nonnegative completely continuous operator;
- 2) for some natural number n ,

$$\sum_{\substack{j, k=n \\ j \neq k}}^{\infty} \mu_{\lambda_j} \mu_{\lambda_k} \frac{\operatorname{Im} \lambda_j \operatorname{Im} \lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} < \infty;$$

- 3)

$$\overline{\lim}_{\substack{j, k \rightarrow \infty \\ j \neq k}} \frac{(\mu_{\lambda_j} - 1) \operatorname{Im} \lambda_j}{|\lambda_j - \bar{\lambda}_k|} < \frac{1}{2\sqrt{2}},$$

then the sequence of the corresponding root subspaces $\{\mathfrak{E}_{\lambda_k}\}_1^\infty$ is a Bari basis of its closed linear span \mathfrak{E} , and the union of the orthonormal bases of the subspaces \mathfrak{E}_{λ_k} ($k = 1, 2, \dots$) forms a Riesz basis of the subspace \mathfrak{E} .

Let us note that conditions 2) and 3) are satisfied if T is an operator of finite trace and $|\lambda_n - \lambda_m| \geq \delta > 0$ ($n \neq m$). These conditions are also satisfied if the eigenvalues λ_n and the corresponding orders μ_{λ_n} have the following asymptotics:

$$\lambda_n = n^\theta (C + r_n + s_n), \quad \mu_{\lambda_n} = O(n^\delta),$$

where C is a real number different from zero; $\{r_n\}_1^\infty$ is a sequence of real numbers converging to zero; $|s_n| = O(n^{-2-\delta})$, $\theta \geq 1$, and $\delta \geq 0$.

We also note that Theorem 4 remains valid if condition 1) is replaced by one of the following conditions:

- 1') A is a bounded dissipative operator, and all nonreal eigenvalues λ_n have finite order and none of them is a limit point of the sequence $\{\lambda_n\}_1^\infty$;

1'') A is a closed dissipative operator, and all nonreal eigenvalues λ_n have finite order and are isolated points of the spectrum.

Theorem 5. *If conditions 1) and 3) of Theorem 4 are satisfied, while condition 2) is replaced by the following stronger condition:*

$$2') \quad \sum_{\substack{j,k=n \\ j \neq k}}^{\infty} \min(\nu_{\lambda_j}, \nu_{\lambda_k}) \mu_{\lambda_j} \mu_{\lambda_k} \frac{\operatorname{Im} \lambda_j \operatorname{Im} \lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} < \infty,$$

then the union of the orthonormal bases of the subspaces \mathfrak{E}_{λ_k} ($k = 1, 2, \dots$) is a Bari basis of its closed linear span.

We note that Theorems 4 and 5 remain true if condition 1) is replaced by the following:

1''') A is a bounded dissipative operator, and all nonreal eigenvalues λ_n have finite multiplicity.

Let us observe that, although it would be more natural to choose in each root subspace a basis consisting of vectors forming Jordan chains, with such a choice already in the case $\mu_{\lambda_k} \geq 3$ ($k = 1, 2, \dots$) no conditions imposed on the numbers $\lambda_k, \mu_{\lambda_k}, \nu_{\lambda_k}$ ($k = 1, 2, \dots$) are sufficient to ensure that the sequence obtained by uniting these chains is a basis of its closed linear span. More precisely, whatever the sequences of complex numbers $\{a_k\}_1^\infty$ ($\operatorname{Im} a_k > 0$) tending to zero and the sequences of natural numbers $\{b_k\}_1^\infty$ and $\{c_k\}_1^\infty$, $c_k \geq b_k \geq 3$, one can construct such a completely continuous dissipative operator A that for it $\lambda_k = a_k$, $\mu_{\lambda_k} = b_k$, $\nu_{\lambda_k} = c_k$, and it is impossible to choose a basis of the closed linear span of the subspaces \mathfrak{E}_{λ_k} ($k = 1, 2, \dots$) which is the union of Jordan chains.

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CITED LITERATURE

1. B. R. Mukminov, *DAN*, **99**, No. 4 (1954).
2. N. K. Bari, *DAN*, **54**, No. 5, 383 (1946).
3. I. M. Glazman, *UMN*, **13**, issue 3, 179 (1958).

4. M. K. Fage, *DAN*, **73**, No. 5, 895 (1950); **74**, No. 6, 1053 (1950).
5. I. M. Gelfand, *Scientific Notes of Moscow State University*, issue 148, 4, 224 (1951).
6. M. G. Krein, M. A. Krasnosel' skii, D. P. Mil' man, *Collected Papers of the Institute of Mathematics, Academy of Sciences of the Ukrainian SSR*, No. 11, 97 (1948).
7. N. K. Bari, *Mathematical Collection*, **14** (56), No. 1-2, 51 (1944).
8. M. G. Krein, *UMN*, **12**, issue 3, 333 (1957).

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