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Abstract

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MATHEMATICAL PHYSICS

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INVESTIGATION OF EXACT SOLUTIONS OF NONSTATIONARY DIFFRACTION PROBLEMS IN THE NEIGHBORHOOD OF SLIDING FRONTS

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In the present note, by the method of separating out the nonanalytic parts ([1]), we investigate the properties of the wave field $u(P,t)$ (P is a point of space, t is time) in the region of geometrical shadow arising in a two-component acoustic medium with cylindrical or spherical interfaces.

1. Let $r = R$ be the equation of the interface in cylindrical (r, z, φ) or spherical (r, θ, φ) coordinates. We shall assume that the wave field $u(P,t)$ is produced by the action at the point Q of a linear $(r = R_1,\ \varphi = 0)$ or point $(r = R_1,\ \theta = \varphi = 0)$ source (respectively for the cases of a cylindrical or spherical boundary) of intensity $a(t) = t^\alpha \varepsilon(t)$; $\varepsilon(t)$ is the unit Heaviside function. The field $u(P,t)$ is the solution of the problem

$$\Delta u - c^{-2}(r)u_{tt} = a(t)\delta(Q), \qquad u|_{t=0} = u_t|_{t=0} = 0,$$

$$\mu_1 u|_{r=R-0} = \mu_2 u|_{r=R+0}, \qquad u_r|_{r=R-0} = u_r|_{r=R+0}, \tag{1}$$

in which $c(r) = c_1$ for $r < R$ and $c(r) = c_2$ for $r > R$, where c_1, μ_1, c_2, μ_2 are constant propagation velocities and densities of the inner $(r < R)$ and outer $(r > R)$ media; $\delta(Q)$ is the delta function.

2. Let l be an extremal of the integral $\tau = \int_Q^P c^{-1}\, ds$ from the class of piecewise continuous curves, one or several of whose links are situated on the interface between the media on the side of the greater value of the velocity. We shall call the *sliding front* the geometric locus of points P for which the equality $\tau = 1$ holds, under the condition that the integral is taken along an extremal l.* The distance of an arbitrary point P from the sliding front, computed along the extremal l passing through the point P, will be denoted by γ. The link of this extremal situated on the interface will be called the *sliding arc* δ. At points of the sliding front $\gamma = 0$; in front

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of the sliding front $\gamma < 0$, behind it $\gamma > 0$; for all points of the region of geometrical shadow $\delta > 0$.

3. On the basis of an investigation of exact solutions ($^{2-4}$) of problems with cylindrical or spherical boundaries for the nonstationary wave field $u(P,t)$, in a neighborhood of a sliding front (outside other fronts) one can obtain the representation

$$u(P,t) = \operatorname{Re}\left[\sum_{s=1}^{\infty}\sum_{n=m}^{\infty} T_s[i\nu(n)] + f(P,t)\right], \tag{2}$$

* For example, in the case of a cylindrical boundary with $c_2 > c_1$, the sliding fronts corresponding to waves that have undergone different numbers of reflections are cylindrical surfaces parallel to the z-axis, whose directrices are the evolvents of the circle $r = R$.

in which $f(P,t)$ is a function regular in a neighborhood of $\gamma = 0$. In formula (2) the following notation has been introduced: $T_s(i\nu)$ is the product of the residue of a certain combination of cylindrical functions at the root $z_s(i\nu)$ of one of the equations

$$H_\nu^{(2)}(-iz) = 0, \tag{3}$$

$$H_\nu^{(2)\prime}(-iz) = 0, \tag{4}$$

$$H_\nu^{(1)}(-iz)H_\nu^{(2)\prime}(-iaz) - \frac{b}{a}H_\nu^{(1)\prime}(-iz)H_n^{(2)}(-iaz) = 0, \quad a = \frac{c_1}{c_2}, \quad b = \frac{\mu_2}{\mu_1}, \tag{5}$$

by $e^{\mp i\nu\varphi}$ in the case of a cylindrical boundary and by

$$\frac{\nu}{2}\left[P_{\nu-1/2}(\cos\varphi) \pm \frac{2i}{\pi}Q_{\nu-1/2}(\cos\varphi)\right]$$

in the case of a spherical boundary. In equations (3)–(5), $H_\nu^{(1,2)}(\xi)$ are Hankel functions of the first and second kinds; $P_\nu(\xi)$ and $Q_\nu(\xi)$ are Legendre functions*. Equation (5) corresponds to conditions (1), and equations (3) and (4) to the same conditions for $\mu_1 = 0$ and $\mu_1 = \infty$. For a cylindrical boundary $\nu(n) = n$, for a spherical one $\nu(n) = n + 1/2$. Finally, $m = 2s - 1$ in the case of equation (4) and $m = 2s$ in the cases (3) and (5).

4. For large $|\nu|$ in the sector $|\arg\nu| < \pi/2$ there is the asymptotic representation

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$$T_s(i\nu) = \nu^{-\beta} \exp\left[i\nu\gamma - \varkappa_s(\gamma+\delta)\nu^{1/3}e^{i\pi/6} - \frac{i\pi\beta}{2}\right] \times$$

$$\times \left[\sum_{k=0}^{3N-1} a_k^{(s)}(P,t)(\nu e^{i\pi/2})^{-k/3} + R_{3N}^{(s)}(P,t,i\nu)(i\nu)^{-N}\right] \tag{6}$$

with real $a_k^{(s)}$. In equality (6) the numbers $\varkappa_s$ $(s = 1, 2, 3, ...)$ are the zeros of the Airy function $W(2^{1/3}\varkappa e^{i\pi/3})$ or of its derivative, $\varkappa_s^{3/2}$ for $s \gg 1$ being proportional to s; β is a constant depending on the constant α characterizing the function $a(t)$. It can be shown that the coefficients $a_k^{(s)}(P,t)$ and the remainder term $R_{3N}^{(s)}(P,t,i\nu)$ are regular functions of P and t in a neighborhood of the creeping front that contains no boundaries of geometrical shadow and no interface of media. One can also prove the inequalities

$$|a_k^{(s)}(P,t)| < c\Gamma(k/3+1)\sigma^k, \qquad |R_{3N}^{(s)}(P,t,i\nu)| < CN!\sigma^{3N}, \tag{7}$$

in which c, C, σ are constants independent of k and N, and $\Gamma(x)$ is the gamma function.

5. Let us separate out the nonanalytic part of the series

$$\mathrm{Re}\sum_{n=m}^{\infty} T_s[i\nu(n)]. \tag{8}$$

Using the properties of the coefficients $a_k^{(s)}$ and applying Watson' s summation method, one can obtain for (8), when $-\sigma^{-3} < \gamma < \sigma^{-3}$, a representation in the form of the sum of a convergent series and a regular function $g_s(P,t)$ (see theo-

* For example, in the case of a cylindrical boundary, for $r > R > R_1$, $a(t) = \varepsilon(t)$, for the creeping front corresponding to the first refracted wave,

$$T_s(in) = \frac{ibH_n^{(1)}(-iR_1R^{-1}z)}{2\pi^2 H_n^{(1)}(-iz)} \times$$

$$\times \left.\frac{H_n^{(2)}(-irR^{-1}az)H_n^{(2)}(-iaz)\exp(-in\varphi + c_1R^{-1}tz)}{a^2(1-b)z^2[H_n^{(2)\prime}(-iaz)]^2 + b[n^2(b-1)+(b-a^2)z^2][H_n^{(2)}(-iaz)]^2}\right|_{z=z_s},$$

where z_s is a root of equation (5).

rem 2 from (1)):

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$$\operatorname{Re}\sum_{n=m}^{\infty} T_s[i\nu(n)] = u_s(P,t)\,\varepsilon(\gamma) + g_s(P,t), \tag{9}$$

$$u_s(P,t) = \frac{3}{2}\sum_{k=0}^{\infty} a_k^{(s)}[\varkappa_s(\gamma+\delta)]^{3\beta-3+k} W_{3\beta-2+k}(p_s), \tag{10}$$

$$W_q(p) = \frac{1}{ip^{q-1}}\int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} t^{-q}\exp\left[p\left(\frac{t^3}{3}-t\right)\right]dt, \qquad p_s \equiv \frac{\varkappa_s^{3/2}(\gamma+\delta)^{3/2}}{\sqrt{3\gamma}}.$$

For $p > 0,\ q > 0$, the function $W_q(p)$ admits the asymptotic representation

$$W_q(p) = \sqrt{\frac{2\pi}{2p\chi+3q}}\,\frac{\exp(-{}^2/_3 p\chi + q/3)}{(p\chi)^{q-1}}\left[1 + O\left(\frac{1}{2p\chi+3q}\right)\right], \tag{11}$$

in which

$$\chi = \sqrt[3]{\frac{q}{2p} + \sqrt{\frac{q^2}{4p^2} - \frac{1}{27}}} + \sqrt[3]{\frac{q}{2p} - \sqrt{\frac{q^2}{4p^2} - \frac{1}{27}}} \geqslant 1.$$

From (11) and (7) it follows that, for $0 \leqslant \gamma < \sigma^{-3}$ and fixed $\delta > 0$, the series (10), as well as the series obtained from it by termwise differentiation with respect to γ any number of times, converge uniformly with respect to γ. Using (11), we obtain

$$u_s(P,t) = \frac{3}{2}\sum_{k=0}^{N} a_k^{(s)}[\varkappa_s(\gamma+\delta)]^{3\beta-3+k} W_{3\beta-2+k}(p_s) + e^{-{}^2/_3 p_s} O\left(p_s^{5/2-3\beta-N}\right), \tag{12}$$

whence, for $N = 0$,

$$u_s(P,t) = \frac{\sqrt{3\pi}\,a_0^{(s)}}{2\sqrt{\gamma}}\left[\frac{3\gamma}{\varkappa_s(\gamma+\delta)}\right]^{-3/2\beta-3/4}\exp\left[-\frac{2}{3}\frac{(\varkappa_s\delta)^{3/2}}{\sqrt{3\gamma}}\right]\left[1 + O\left(\frac{\sqrt{\gamma}}{(\varkappa_s\delta)^{3/2}}\right)\right]. \tag{13}$$

Formula (12) establishes the behavior of $u_s(P,t)$ near $\gamma = 0$. From formula (13), in view of the uniform convergence with respect to γ of the series obtained by termwise differentiation with respect to γ of the series (10), it follows that the function $u_s(P,t)$, with all derivatives, is continuous in a neighborhood of the sliding front $\gamma = 0$. For $\gamma = 0$, the functions $u_s(s \gg 1)$ have singularities of

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the form $\exp(-a_1 s\delta\gamma^{-1/2})$, $a_1 > 0$. Thus, as s increases, the contribution of the terms of the outer series (2) to the singularity of the wave field at the sliding front decreases exponentially.

6. In the cases $\mu_1 = 0$ or $\mu_1 = \infty$, using the properties of $H_\nu^{(2)}(-iz)$ and $T_s(i\nu)$ for imaginary ν *, one can obtain another representation for the series (8):

$$\operatorname{Re}\sum_{n=m}^{\infty} T_s[i\nu(n)] = -\frac{3}{2}\int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} T_s(\xi^3)\xi^2\, d\xi\cdot\varepsilon(\gamma) + \operatorname{Re} g_s(P,t), \qquad (14)$$

in which $g_s(P,t)$, as well as $\sum_{s=1}^{\infty} g_s(P,t)$, are regular in a neighborhood of $\gamma = 0$.

Thus the field $u(P,t)$ is split into the analytic function $f(P,t)+\sum_{s=1}^{\infty} g_s(P,t)$ and the function $-\frac{3}{2}\sum_{s=1}^{\infty}\int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} T_s(\xi^3)\xi^2 d\xi\cdot\varepsilon(\gamma)$, which has a singu-

* It can be shown that equations (3) and (4), for $\operatorname{Re}\nu = 0$, have an infinite set of real roots on the interval $0 < z < |\operatorname{Im}\nu|$, with a point of accumulation $z = 0$. These roots have no roots in the domain $|\arg z| < \pi/2$. For $\operatorname{Re}\nu = 0$ and $z > 0$, the function $H_\nu^{(2)}(-iz)$ is imaginary and is expressed in terms of the tabulated function (5). The expressions $T_s(i\nu)$ in cases (3) and (4) are also imaginary for imaginary ν.

···ness for $\gamma = 0$. If the sliding front under consideration propagates through a medium that has not yet been disturbed, then the analytic part of the field is identically equal to zero.

7. Formula (12) makes it possible to compute approximately the values $u(P,t)$ in the neighborhood of a sliding front. We give the values of several first coefficients $a_k^{(s)}$ in various diffraction problems*.

A. Cylindrical boundary, $c_2 > c_1,\ r > R,\ R_1 > R,\ a(t) = t^\alpha\varepsilon(t)$.

$$\gamma = R^{-1}\left(c_2 t - \sqrt{R_1^2 - 1} - \sqrt{r^2 - R^2}\right) - \delta,$$

$$\delta = \varphi - \arccos RR_1^{-1} - \arccos Rr^{-1}, \qquad \beta = \alpha + \frac{5}{3},$$

$$a_0^{(s)} = -\frac{\Gamma(\alpha+1)}{2^{1/3}\pi}\left(\frac{R}{c_2}\right)^\alpha \frac{R\,|W'(2^{1/3}\chi_s e^{i\pi/3})|^{-2}}{\sqrt[4]{(R_1^2 - R^2)(r^2 - R^2)}}\exp\left(-\frac{\delta a}{b\sqrt{1-a^2}}\right),$$

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$$a_1^{(s)} = \frac{\chi_s^2}{10}\left[3(\gamma+\delta) - 5R\left(\frac{1}{\sqrt{R_1^2 - R^2}} + \frac{1}{\sqrt{r^2 - R^2}}\right)\right] a_0^{(s)}.$$

B. Cylindrical boundary, $c_2 > c_1,\ r > R > R_1,\ a(t) = t^\alpha \varepsilon(t)$.

$$\gamma = (aR)^{-1}\left(c_1 t - R\sqrt{1-a^2} + \sqrt{R_1^2 - a^2R^2} - a\sqrt{r^2 - R^2}\right) - \delta,$$

$$\delta = \varphi - \arccos(Rr^{-1}) + \arccos(a^{-1}RR_1^{-1}) - \arccos a^{-1}, \qquad \beta = \alpha + \frac{11}{6},$$

$$a_0^{(s)} = \frac{a\Gamma(\alpha+1)(Rc_2^{-1})^\alpha R \exp(\delta a/b\sqrt{1-a^2})}{2^{7/6}\pi b\,|W'(2^{1/3}\chi_s e^{i\pi/3})|\sqrt[4]{(R_1^2 - a^2R^2)(r^2-R^2)(1-a^2)}},$$

$$a_1^{(s)} = \frac{\chi_s^2}{10}\left[3(\gamma+\delta) - 5\left(\frac{a}{\sqrt{1-a^2}} - \frac{aR}{\sqrt{R_1^2 - a^2R^2}} + \frac{R}{\sqrt{r^2-R^2}}\right)\right] a_0^{(s)}.$$

C. A plane wave $u = \varepsilon[t - c_2^{-1}(R - r\cos\varphi)]$ is incident on a **cylindrical boundary** ($\mu_1 = \infty$).

$$\gamma = R^{-1}\left(ct - R - \sqrt{r^2 - R^2}\right) - \delta, \qquad \delta = \varphi - \pi/2 - \arccos Rr^{-1}, \qquad \beta = \frac{7}{6},$$

$$a_0^{(s)} = \frac{1}{2\sqrt{\chi_s}}\left(\frac{r^2}{R^2} - 1\right)^{-1/4}, \qquad a_1^{(s)} = \chi_s^2\left(\gamma + \delta - \frac{R}{\sqrt{r^2-R^2}}\right) a_0^{(s)}.$$

One may conjecture that also in more complicated problems (variable propagation speeds, an interface boundary–sufficiently smooth cylindrical surfaces of arbitrary cross-section), in a neighborhood of sliding fronts the wave field will have a singularity of the same type as the singularities of the functions $u_s(P,t)$.

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* In works ([6–8]), in some diffraction problems for one-component media with cylindrical and spherical interfaces, analogous formulas (13) were obtained. None of these works contains formulas of the form (9), (10), or (14), as a result of which the question of further approximations and of the behavior of the field as a whole in the neighborhood of sliding fronts remained unclear.

Note: Figure translations are in progress. See original paper for figures.

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