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Abstract

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On Embedding Theorems

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Many results in the area of embedding theorems can be obtained by direct integral estimates with the aid of the simplest analytic means. The proofs belonging here, while possessing great internal generality, at the same time differ substantially from other known methods pursuing the same aims. It is therefore natural to speak of them as a single method, which we shall call the direct method.

The distinctive features of the direct method consist approximately in the following:

1. The introduction into intermediate calculations of "free parameters," which are subsequently chosen so as to satisfy certain relations. Free parameters may be introduced by applying Hölder' s inequality with an undetermined exponent, by representing $|u|^q$ in the form $|u|^\alpha |u|^{q-\alpha}$ or $(|u|^\alpha)^{q/\alpha}$, and in other ways.
2. The systematic use of the notions of dimension and differential order, discussed below.
3. The use of simple integral representations, most often not of the given function itself, but of its powers.

By the present time, the direct method has yielded, besides new proofs of the generally known Sobolev-Kondrashov theorems, a number of inequalities containing more than two functional norms. Some of them were proved by other, more complicated methods in papers ($^{1-3}$). The direct method is especially well adapted for obtaining results of this kind. The number of independent exponents in the result is determined by the number of free parameters introduced in the course of the proof and by the relations imposed on them. Any result obtained by the direct method is usually easily generalized by introducing additional free parameters.

Recently Lu Wen-tian extended the direct method to embedding theorems concerning spaces of functions having different properties with respect to different groups of variables.

We note that the proofs of S. L. Sobolev' s theorems given by Gagliardo (^{4}) are undoubtedly close to what we have called the direct method. (However, in paper ([4]), instead of free parameters, their final values are substituted at once.)

In the present note we formulate for the first time certain notions whose significance became clear in connection with the systematic application of the direct method, and we present new results obtained by this method.

Let $F[u(x)]$ be a homogeneous functional, invariant under translations of the argument $u(x)$, defined on a set of finite functions whose diameter of the domain of finiteness does not exceed a given number d.

Definition 1. We shall say that $F[u(x)]$ has dimension $\varkappa$ if

$$F[u(\lambda x)] = \lambda^{-\varkappa} F[u(x)].$$

Definition 2. We shall say that $F[u(x)]$ is p-additive if

$$F\left[\sum_{i=1}^{N} u(x - x_i)\right] = N^{1/p} F[u(x)], \qquad 1 \leq p \leq \infty,$$

where it is assumed that,

that the domains of finiteness of the separate terms in the sum $\sum_{i=1}^{N} u(x - x_i)$ are separated from one another by a distance exceeding the diameter of the latter domains a prescribed number of times, while the diameter of the domain of finiteness of the whole sum does not exceed d.

Definition 3. We shall call the **differential order of a functional** with the properties indicated in the preceding definitions the number

$$l = \frac{n - \chi p}{p},$$

where n is the dimension of the space.

It is easy to see that our definitions are applicable to the norms in L_q, C, W_p^l, $\operatorname{Lip}\alpha$, to the geometric means of these norms, and to a number of other expressions.

Lemma. *If F_1 and F_2 have dimensions respectively equal to χ_1 and χ_2, and differential orders l_1 and l_2, then in order that the estimate*

$$F_1[u(x)] \leq C F_2[u(x)], \tag{1}$$

with C independent of $u(x)$, hold, it is necessary that $\chi_1 \geq \chi_2$ and $l_1 \leq l_2$.

This lemma makes it possible in a large number of cases, without resorting to the construction of examples, to prove the necessity of one or another restriction in embedding theorems. In particular, in all cases known to us the limiting exponents are characterized by the fact that substituting them into (1) leads to the equality $\chi_1 = \chi_2$, whence it follows that they can be changed only in one, "unfavorable," direction. By requiring that $\chi_1 = \chi_2$, one can guess limiting exponents, which is nontrivial if the estimate contains three or more norms. (This also gives an excellent way of remembering the limiting exponents.) At the same time, the use of the notions of dimension and differential order makes it possible to carry out a "selection of means" for the proof, to clarify the possibilities of various methods, and so on. Thus, all paths connected with a "loss of dimension" are obviously "bad," and therefore the dimensions of all expressions introduced into intermediate calculations must lie between the maximal and minimal dimensions of the norms entering into the expected result. To obtain results of the Gagliardo–Nirenberg inequality type, paths connected with an increase of the differential order in some link of the reasoning, i.e. the use of the usual embedding theorems, are "bad."

We pass to concrete results. Let x, x', and x'' denote, respectively, points of the n-dimensional space E_n, of the s-dimensional hyperplane E_s, and of the $(n-s)$-dimensional orthogonal complement E_{n-s} to it. Denote by $W_p^{l,n-s}$ the space of functions defined in some domain $\Omega \subset E_n$ satisfying the cone condition and having derivatives of order l with respect to the variables x'', summable with exponent p in Ω. The norm in $W_p^{l,n-s}$ is defined as

$$\left\{ \int_\Omega \sum_{k_1+\cdots+k_{n-s}=l} \left(\frac{\partial^l u}{\partial {x''_1}^{k_1} \ldots \partial {x''_{n-s}}^{k_{n-s}}} \right)^p dx + \int_\Omega |u|^p dx \right\}^{\frac{1}{p}}.$$

Theorem 1. *If the conditions $pl > n-s$, $lq > n-s$, $q > p > 1$ are satisfied, then*

$$\|u\|_{L_q(E_s \cap \Omega)} \le C \|u\|^{\frac{n-s}{lq}}_{W_p^{l,n-s}(\Omega)} \|u\|^{\frac{lq-n+s}{lq}}_{L_{\frac{p(lq-n+s)}{lp-n+s}}(\Omega)}. \tag{2}$$

We shall sketch the proof of this theorem, restricting ourselves for simplicity to finite functions.

The inequality

$$\left\{ \int_\Omega \frac{|u|^q}{r^h} \, dx \right\}^{\frac{1}{q}} \le C \|u\|^{\frac{h}{lq}}_{W_p^{(l)}} \|u\|^{\frac{lq-h}{lq}}_{\frac{lp(lq-h)}{lp-h}}, \qquad h \le lq, \ h < n, \tag{3}$$

holds; it is especially simple to prove when $l = 1$ and $h > 1$. In this case one must represent $\left(\frac{1}{r}\right)^h$ in the form $\frac{1}{1-h} \frac{d}{dr} \left(\frac{1}{r}\right)^{h-1}$, perform integration by parts in the left-hand side, and apply Hölder' s inequality. From this special case of (3) and from inequality (4) it is easy to derive inequality (3) in full generality. We note that (3) also follows from a more general inequality of V. P. Il' in [5].

From (3) the assertion of the theorem is easily obtained in the special case when $p(l-1) < n-s$. Indeed, denoting by ρ the distance in the subspace E_{n-s}, we shall have

$$\int_{E_s} |u^q(x', x'')|\, dx' \leq C \int_{E_s} dx' \int_{E_{n-s}} \frac{|u^{q-1}(x', y) u_\rho(x', y)|}{\rho_{x''y}^{n-s-1}}\, dy \leq$$

$$\leq C \int_{E_s} dx' \left\{ \int_{E_{n-s}} \frac{|u_\rho^p|}{\rho^{(l-1)p}}\, dy \right\}^{\frac{1}{p}} \left\{ \int_{E_{n-s}} \frac{|u|^{\frac{(q-1)p}{p-1}}}{\rho^{\frac{(n-s-l)p}{p-1}}}\, dy \right\}^{\frac{p-1}{p}}.$$

Hence, applying inequality (3) and Hölder' s inequality, we obtain the assertion of the theorem. The general case is reduced to this special one by applying the Gagliardo-Nirenberg inequality [3,4], according to which

$$\|u\|_{W_{p_m}^{(m)}} \leq C \|u\|_{W_{p_l}^{(l)}}^{\frac{m}{l}} \|u\|_{L_{p_0}}^{\frac{l-m}{l}}, \qquad \frac{l}{p_m} = \frac{m}{p_l} + \frac{l-m}{p_0}, \tag{4}$$

For this we apply Theorem 1, introducing into consideration the norm in $W_{p_m}^{(m)}$ with undetermined m and p_m, assuming that $(m-1)p_m < n-s$. Then we use the Gagliardo-Nirenberg inequality and, by choosing p_m, achieve that both norms entering into the estimate in L_q coincide. In this process m drops out of the result (which can be foreseen from dimensional considerations), and it is possible to prove that it can always be taken equal to an integer such that all the preceding computations are justified.

On the basis of our lemma, the necessity of the first two conditions of the theorem is easily established, and here one must distinguish the dimensions with respect to E_s and with respect to E_{n-s}. On the other hand, the third condition can in a number of cases be weakened. Finally, we note that, taking ρ so that the total dimensions of both factors in (2) coincide, we arrive at S. L. Sobolev' s theorem on the embedding of $W_p^{(l)}(E)$ in $L_q(E_s)$.

Theorem 2. If the conditions

$$1 \geq \alpha \geq \frac{r-h}{l-h}; \qquad p_r,\ p_l > 1;$$

$$\frac{1}{p_r} > \frac{r-h}{n} + \alpha \left(\frac{1}{p_l} - \frac{l-h}{n} \right),$$

hold, then

$$\|u\|_{W_{p_r}^{(r)}} \leq C \|u\|_{W_{p_l}^{(l)}}^{\alpha} \|u\|_{\mathrm{Lip}\, h}^{1-\alpha}.$$

If, in addition, $lp_l < n$,

$$\alpha > \frac{r(n - lp_l) + lp_{lh}}{l\,[hp_l + n - lp_l]},$$

then the theorem is also true for the limiting value of p_r. The necessity of the condition $\alpha \geq \frac{r-h}{l-h}$ follows from the lemma.

Theorems 1 and 2 contain assertions on the nontrivial embedding of the intersection of certain two spaces into a third, when neither of the first two is contained in the third as a whole.

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Note: Figure translations are in progress. See original paper for figures.