# MATHEMATICS

A. L. ONISHCHIK

1960

SovietRxiv

**Abstract**

**Full Text**

*MATHEMATICS*

A. L. ONISHCHIK

# ON COMPACT LIE GROUPS TRANSITIVE ON CERTAIN MANIFOLDS

*(Presented by Academician P. S. Aleksandrov, 17 VI 1960)*

In the works ([1,2]), Montgomery, Samelson, and Borel determined all connected compact Lie groups acting transitively on spheres. In the present work the problem of finding all connected compact Lie groups transitive on a given manifold is solved for certain other classes of homogeneous manifolds*. At the same time, a general topological method is given for approaching this problem. In particular, a new proof of the results of Montgomery—Samelson—Borel is obtained.

1. We shall denote by $H(X)$ the cohomology algebra of the space $X$ with real coefficients, and by $f^*$ the homomorphism of cohomology algebras associated with the continuous map $f$. The Poincaré polynomial of a graded space $X$ or of a topological space $X$ will be denoted by $P(X, t)$.

Let $\mathfrak{G}$ be a connected compact Lie group, $\mathfrak{U}$ its closed subgroup, $\mathfrak{U}_0$ the connected component of the identity in the group $\mathfrak{U}$, and $i : \mathfrak{U}_0 \to \mathfrak{G}$ the embedding. Denote by $P$ and $Q$ the spaces of primitive elements of the algebras $H(\mathfrak{G})$ and $H(\mathfrak{U}_0)$. Then $i^*(P) \subset Q$. Hence there exist graded spaces $P_1, P_2 \subset P$ and $Q_1, Q_2 \subset Q$ such that $P = P_1 \oplus P_2$, $Q = Q_1 \oplus Q_2$, $i^*P_1 = 0$, and $i^*$ maps $P_2$ isomorphically onto $Q_2$. The basic topological fact used in the present work is that the graded spaces $P_1$ and $Q_1$ are topological invariants of the manifold $X = \mathfrak{G}/\mathfrak{U}$, i.e. they do not depend on the choice of a compact group $\mathfrak{G}$ transitive on this manifold.

This fact follows from the following theorem, whose proof is based on the results of ([3,4]):

**Theorem 1.** *The graded spaces $P_1, Q_1$, and also the rational fraction*

$$\frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}_0, t)}$$

*are topological invariants of the manifold $X = \mathfrak{G}/\mathfrak{U}$. More precisely, if $r_k$ is the rank of the homotopy group $\pi_k(X)$, then*

$$P(P_1, t) = \sum_{k=1}^{\infty} r_{2k-1} t^{2k-1}, \qquad P(Q_1, t) = \sum_{k=1}^{\infty} r_{2k} t^{2k-1},$$

$$\frac{P(\mathfrak{G},t)}{P(\mathfrak{U}_0,t)}=\prod_{k=1}^{\infty}\left(1+t^{2k-1}\right)^{r_{2k-1}-r_{2k}}.$$

The invariance of the fraction

$$\frac{P(\mathfrak{G},t)}{P(\mathfrak{U}_0,t)}$$

was already noted in ([5]). In the case where $\mathfrak{U}$ is a connected subgroup of maximal rank in $\mathfrak{G}$, it follows from Hirzebruch' s formula.

Let $X$ be some topological space. We shall denote by $r(X)$ and call the *rank* of the space $X$ the sum of the ranks of all groups $\pi_{2k-1}(X)$ $(k=1,2,...)$, if this sum is finite. If $X=\mathfrak{G}/\mathfrak{U}$–

* All transformation groups are assumed to be effective.

homogeneous space of the compact group $\mathfrak{G}$, then from Theorem 1 it follows that $r(X)=\dim P_1$. In particular, $r(\mathfrak{G})$ is equal to the ordinary rank of the group $\mathfrak{G}$.

**2.** In this section we study homogeneous spaces of compact Lie groups having rank 1, and find all compact Lie groups transitive on these manifolds.

**Theorem 2.** *Let a connected compact Lie group $\mathfrak{G}$ act transitively on a manifold $X$ of rank $r$. Then there exists a normal divisor $\mathfrak{G}'\subset\mathfrak{G}$, locally isomorphic to a direct product of no more than $r$ simple groups and transitive on $X$. In particular, if $r=1$, then $\mathfrak{G}'$ is a simple normal divisor.*

*Furthermore, if $\mathfrak{G}''$ is a normal divisor of the group $\mathfrak{G}$, complementary to $\mathfrak{G}'$, then $r(\mathfrak{G}'')\leq r$.*

From Theorem 2 it is clear that, in order to find all homogeneous spaces of rank 1, it suffices to consider homogeneous spaces of simple compact Lie groups $\mathfrak{G}$. If $\mathfrak{G}$ is commutative, then $X=S^1$, the circle. If $\mathfrak{G}$ is simple and noncommutative, then locally isomorphic homogeneous spaces of the group $\mathfrak{G}$ have equal ranks. Therefore one must enumerate all pairs $(G,U)$, where $G$ is a simple noncommutative compact Lie algebra and $U$ is its subalgebra, that give homogeneous spaces of rank 1.

**Theorem 3.** *All pairs $(G,U)$ that give homogeneous spaces of rank 1 are listed in Table 1, in which the simply connected homogeneous spaces $X$ corresponding to these pairs are also indicated. Two spaces $X$ are homeomorphic if and only if they are denoted in the same way.*

**Table 1**

| $G$ | $U$ | $i$ | $X$ | $G$ | $U$ | $i$ | $X$ |
|---|---|---|---|---|---|---|---|
| $A_n$ | $A_{n-1}$ | $\varphi_1\dot{+}N$ | $S^{2n+1}$ | $C_n$ $(n>1)$ | $C_{n-1}$ | $\varphi_1\dot{+}2N$ | $S^{4n-1}$ |

| $G$ | $U$ | $i$ | $X$ | $G$ | $U$ | $i$ | $X$ |
|---|---|---|---|---|---|---|---|
| $A_n$ | $A_{n-1} \oplus T$ | $\varphi_1 \dot{+} N$ | $PC^n$ | $C_n$ $(n>1)$ | $C_{n-1} \oplus T$ | $\varphi_1 \dot{+} 2N$ | $PC^{2n-1}$ |
| | | | | $C_n$ $(n>1)$ | $C_{n-1} \oplus C_1$ | $\varphi_1 \dot{+} 2N$ | $PK^{n-1}$ |
| $A_2$ | $A_1$ | $\overset{2}{\circ}$ | $SU(3)/SO(3)$ | $D_n$ | $B_{n-1}$ | $\varphi_1 \dot{+} N$ | $S^{2n-1}$ |
| $B_n$ $(n>1)$ | $B_{n-1}$ | $\varphi_1 \dot{+} 2N$ | $V_{2n+1,2}$ | $F_4$ | $B_4$ | | $PO^2$ |
| $B_n$ $(n>1)$ | $B_{n-1} \oplus T$ | $\varphi_1 \dot{+} 2N$ | $G_{2n+1,2}$ | $G_2$ | $A_1^1$ | | $V_{7,2}$ |
| $B_n$ $(n>1)$ | $D_n$ | $\varphi_1 \dot{+} N$ | $S^{2n}$ | $G_2$ | $A_1^1 \oplus T$ | | $G_{7,2}$ |
| $B_4$ | $B_3$ | $\varphi_3 \dot{+} N$ | $S^{15}$ | $G_2$ | $A_1^3$ | | $G_2/A_1^3$ |
| $B_3$ | $G_2$ | $\varphi_2$ | $S^7$ | $G_2$ | $A_1^3 \oplus T$ | | $G_2/A_1^3 \times T$ |
| $B_2$ | $A_1$ | $\overset{4}{\circ}$ | $B_2/A_1^{10}$ | $G_2$ | $A_1^1 \oplus A_1^3$ | | $PO^1$ |
| | | | | $G_2$ | $A_1^{14}$ | | $G_2/A_1^{14}$ |
| | | | | $G_2$ | $A_1^{28}$ | | $G_2/A_1^{28}$ |
| | | | | $G_2$ | $A_2$ | | $S^6$ |

In Table 1, $i$ denotes the embedding $U \to G$, and in the corresponding column (for the classical algebras $G$) the linear representation of the algebra $U$ realizing this embedding is indicated; here the notation of [5] is used. If the algebra $U$ is not simple, then the linear representation of its simple ideal having the greatest rank is indicated. By $A_1^k$ is denoted a subalgebra of type $A_1$ and index $k$ in $G_2$, and by $T$ a one-dimensional Lie algebra. In po-

In the last column the following notation is used: $S^n$ is the $n$-dimensional sphere; $PC^n$, $PK^n$, $PO^n$ are, respectively, the complex, quaternionic, and octonionic projective spaces of dimension $n$; $V_{n,2}$ is the Stiefel manifold of orthonormal 2-frames in $n$-dimensional Euclidean space $R^n$; $G_{n,2}$ is the Grassmann manifold of oriented planes in $R^n$.

Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be two groups of transformations of a manifold $X$. We say that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are **similar** if there exists a homeomorphism $A$ of the manifold $X$ onto itself such that $A\mathfrak{G}_1A^{-1} = \mathfrak{G}_2$.

**Corollary of Theorem 3.** a) *Every connected compact Lie group transitive on $S^n$ is similar to the group $SO(n+1)$ or to one of the following subgroups of it: $SU(m)$, $U(m)$ $(n = 2m-1)$; $Sp(2m)$, $Sp(2m) \times U_1$, $Sp(2m) \times Sp(2)$ $(n = 4m-1)$; $Spin(9)$ $(n = 15)$; $Spin(7)$ $(n = 7)$; $G_2$ $(n = 6)$.*

b) *Every connected compact Lie group transitive on $PC^n$ is similar to the group $SU(n+1)$ or (for $n = 2m-1$) to its subgroup $Sp(2m)$.*

c) *Every connected compact Lie group transitive on $G_{2n+1,2}$, or on the man-*

*ifold of unoriented planes $\widetilde{G}_{2n+1,2}$, is similar to the group $SO(2n+1)$ or (for $n=3$) to its subgroup $G_2$.*

d) *If $n \neq 3$, then every connected compact Lie group transitive on $V_{2n+1,2}$, or on the manifold $\widetilde{V}_{2n+1,2}$ obtained from $V_{2n+1,2}$ by identifying the frames $e_1, e_2$ and $-e_1, -e_2$, is similar to the group $SO(2n+1)$ or $SO(2n+1) \times SO(2)$. Every connected compact Lie group transitive on $V_{7,2}$ or $\widetilde{V}_{7,2}$ is similar to one of the following groups: $SO(7)$, $SO(7) \times SO(2)$, $G_2 \subset SO(7)$, $G_2 \times SU(2)$, $G_2 \times U(1)$.*

e) *Every connected compact Lie group transitive on $G_2/A_1^3$ is similar to the group $G_2 \times SU(2)$ or to one of its subgroups $G_2$, $G_2 \times U(1)$.*

f) *Every connected compact Lie group transitive on $PK^n$, $SU(3)/SO(3)$, $B_2/A_1^{10}$, $PO^2$, $G_2/A_1^3 \times T$, $G_2/A_1^4$, $G_2/A_1^{28}$, $PO^1$, is similar to the groups $Sp(2n+2)$, $SU(3)$, $SO(5)$, $F_4$, $G_2$, respectively.*

Let us also note the following property of manifolds of rank 1.

**Theorem 4.** *Let $X = \mathfrak{G}/\mathfrak{U}$, where $\mathfrak{G}$ is a connected compact Lie group and $\mathfrak{U}$ is its connected closed subgroup. Then $r(X) = 1$ if and only if $H(X)$ is an algebra with one generator.*

3. Using the fact that the Poincaré polynomials of simple compact Lie groups are known, from Theorem 1 one can obtain some general results about simple transitive transformation groups.

**Theorem 5.** *Let $X = \mathfrak{G}/\mathfrak{U}$, where $\mathfrak{G}$ is a simple connected compact Lie group and $\mathfrak{U}$ is its closed subgroup, and let $\mathfrak{G}'$ be a simple compact Lie group transitive on $X$. Denote by $G, U, G'$ the Lie algebras of the groups $\mathfrak{G}, \mathfrak{U}, \mathfrak{G}'$. If $G$ is the algebra $A_n$ or an exceptional algebra, then, as a rule, $G' \cong G$. If $G$ is one of the algebras $B_n, C_n, D_{n+1}$, then, as a rule, $G'$ is also one of these algebras. The only exceptions to this rule are the following cases: the pair $(G, U)$ defines one of the rank-1 manifolds listed in parts a)–d) of the corollary to Theorem 3; $G = A_{2n}$, $U = C_{n-1}$; $G = A_{2n-1}$, $U = C_n$. In the last two cases we have $G' \cong G$ or $G' = A_{2n+1}$, $G' = A_{2n-2}$, respectively.*

**Corollary.** *Let $\mathfrak{G}$ be a simple compact noncommutative Lie group, and let $\mathfrak{T}$ be its maximal torus. Then every simple connected compact Lie group transitive on $\mathfrak{G}$ or on $\mathfrak{G}/\mathfrak{T}$ is similar to the group $\mathfrak{G}$ (we assume that $\mathfrak{G}$ acts on itself by left translations).*

* The assertion concerning spheres is a result of Montgomery–Samelson–Borel. The question of groups transitive on a simply connected manifold having the same integral cohomology as the nonhomogeneous sphere was considered in paper ([6]), where the corresponding special case of Theorem 2 was proved.

4. Let $X$ be some compact manifold. Theorem 5 shows that, for us, the question of when all compact groups transitive on $X$ are simple is an important one. Consider the case when $X$ is a simply connected manifold

with positive Euler characteristic. Then $X$ has the property indicated above if and only if it is not a direct product of two homogeneous spaces. We shall say that the manifold $X$ is **indecomposable** if it is not a direct product of two manifolds of positive dimension. It can be proved, for example, that a simply connected orientable compact manifold of rank 1 is always indecomposable. We now give one more sufficient condition for the indecomposability of a manifold.

A manifold $X$ of dimension $2km$ is called $k$-**symplectic** if there exists an element $\omega \in H^{2k}(X)$ such that $\omega^m \neq 0$.

**Lemma.** Let $X$ be a $k$-symplectic manifold and let $H^i(X) = 0$ $(0 < i < 2k)$, $\dim H^{2k}(X) = 1$. Then $X$ is indecomposable.

1-symplectic manifolds are called **symplectic**. In [7] it is proved that a homogeneous space of a semisimple compact Lie group is symplectic if and only if it is Kähler. Therefore, from the lemma and from Theorem 5 the following theorem follows.

**Theorem 6.** Let $X = \mathfrak{G}/\mathfrak{U}$, where $\mathfrak{G}$ is a simple compact Lie group. Denote by $G$ and $U$ the Lie algebras of the groups $\mathfrak{G}$ and $\mathfrak{U}$. If $G$ is an algebra of type $A_n$ or an exceptional algebra, and $U$ is the centralizer of its center in $G$, and if the center of the algebra $U$ is one-dimensional, then every connected compact Lie group transitive on $X$ is simple and, as a rule, is similar to the group $\mathfrak{G}$. The only exceptions to this rule are the manifolds $X = PC^{2m-1}$, $G_{7,2}$, $\widetilde{G}_{7,2}$.

In particular, if $X = C_{n,k}$ is the Grassmann manifold of $k$-dimensional subspaces of an $n$-dimensional complex space, then for $k > 1$ every connected compact Lie group transitive on $X$ is similar to the group $SU(n)$.

Moscow State University
named after M. V. Lomonosov

Received
17 VI 1960

## CITED LITERATURE

1. D. Montgomery, H. Samelson, Ann. Math., **44**, 454 (1943).
2. A. Borel, C. R., **230**, 1378 (1950).
3. H. Cartan, J. P. Serre, C. R., **234**, 393 (1952).
4. A. L. Onishchik, Matem. sborn., **44**, 3 (1958).
5. A. L. Onishchik, DAN, **129**, 261 (1959).

6. Y. Matsushima, Nagoya Math. J., **2**, 1 (1951).

7. A. Borel, Proc. Nat. Acad. Sci., **40**, 1147 (1954).

*Note: Figure translations are in progress. See original paper for figures.*