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Abstract

Full Text

Mathematics

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On the Question of Solving an Integral Equation by Means of Series

(Presented by Academician S. N. Bernstein, 19 IV 1960)

In the present note a method is proposed for constructing the resolvent of a symmetric real kernel by means of bilinear series converging uniformly in both variables.

This method is based on the following lemma:

Lemma. If a symmetric real kernel $K(x, s)$ ($a \leq x, s \leq b$) is positive and continuous, and if $\Phi_i(x)$ ($a \leq x \leq b$; $i = 1, 2, \dots$) are continuous real functions such that the kernels

$$K_n(x, s) = K(x, s) - \sum_{i=1}^n \Phi_i(x)\Phi_i(s)$$

are positive, then the sequence of kernels $K_n(x, s)$ converges uniformly in the square $a \leq x, s \leq b$.

The proof of this lemma is obtained by the same device that was used in work ⁽¹⁾ in proving a closely related proposition. Therefore we shall dwell here on the construction of the functions $\Phi_i(x)$ satisfying the conditions of the lemma. To this end, let us note that a kernel $K(x, s)$ possessing the properties enumerated in the lemma can be represented (and, moreover, in various ways) in the form

$$K(x, s) = \int_I H(x, t)H(s, t) d\tau(t), \quad (1)$$

where I is some interval, $\tau(t)$ ($t \in I$) is a nondecreasing function, and $H(x, t)$ belongs to \mathcal{L}_τ^2 for every x from $[a, b]$.

Next take a sequence of real functions $p_i(t) \in \mathcal{L}_\tau^2$ ($i = 1, 2, \dots$), among which there are no linear dependencies, and from them construct the functions

$$\varphi_i(x) = \int_I H(x, t)p_i(t) d\tau(t) \quad (i = 1, 2, \dots), \quad (2)$$

as well as the numbers

$$a_{ij} = \int_I p_i(t)p_j(t) d\tau(t) \quad (i, j = 1, 2, \dots), \quad (3)$$

and with their aid define the functions

$$\Phi_i(x) = \frac{1}{\sqrt{A_{i-1}A_i}} \begin{vmatrix} \varphi_1(x) & \dots & \varphi_i(x) \\ a_{11} & \dots & a_{1i} \\ \dots & \dots & \dots \\ a_{i-1,1} & \dots & a_{i-1,i} \end{vmatrix},$$

where $A_n = |a_{ij}|_{i,j=1}^n$.

It is easy to verify that these functions satisfy the conditions of the lemma, i.e., they are continuous and the kernels $K_n(x, s)$ constructed from them are positive. Moreover, it is easily verified that if the sequence $p_i(t)$ is complete in \mathcal{L}_τ^2 , then $K_n(x, s)$, as $n \rightarrow \infty$, tends uniformly to zero. Therefore, in this case the uniformly convergent expansion

$$K(x, s) = \sum_{i=1}^{\infty} \Phi_i(x)\Phi_i(s). \quad (4)$$

is valid.

Let us note that, by means of Sylvester' s identity for determinants, the kernel $K_n(x, s)$ can be represented in the form

$$K_n(x, s) = \begin{vmatrix} K(x, s) & \varphi_1(x) & \dots & \varphi_n(x) \\ \varphi_1(s) & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ \varphi_n(s) & a_{n1} & \dots & a_{nn} \end{vmatrix}. \quad (5)$$

Let us pass to the resolvents $\Gamma_n(x, s; \lambda)$ and $\Gamma(x, s; \lambda)$ of the kernels $K_n(x, s)$ and $K(x, s)$. In doing so we shall assume that we are dealing with a loaded integral equation with a function of bounded variation $\sigma(x)$ ($a \leq x \leq b$) as the loading function.

The dependence between the resolvents $\Gamma_n(x, s; \lambda)$ and $\Gamma(x, s; \lambda)$, usually called Bateman' s formula ⁽²⁾, is obtained from (5) by replacing $K_n(x, s)$ and $K(x, s)$, respectively, by $\Gamma_n(x, s; \lambda)$ and $\Gamma(x, s; \lambda)$, the functions $\varphi_i(x)$ by the functions

$$\psi_i(x) = \varphi_i(x) + \lambda \int_a^b \Gamma(x, s; \lambda)\varphi_i(s) d\sigma(s) \quad (6)$$

and the numbers a_{ij} by the numbers

$$b_{ij} = a_{ij} + \lambda \int_a^b \varphi_i(x) \psi_j(x) d\sigma(x). \quad (7)$$

Since the uniform convergence to zero of the kernels $K_n(x, s)$ implies the uniform convergence to zero of the resolvents $\Gamma_n(x, s; \lambda)$, the series of the form (4), in which the corresponding substitution of functions and numbers has been made, converges uniformly to $\Gamma(x, s; \lambda)$. The resulting series, of course, is not suitable for computing the resolvent, since its terms are themselves constructed with the help of the resolvent. However, for any real λ distinct from an eigenvalue, this drawback can be eliminated by an appropriate choice of the functions $p_i(t)$.

Indeed, denoting

$$K^*(x, s) = \int_a^b H(t, x) H(t, s) d\sigma(t),$$

put

$$p_i(x) = q_i(x) - \lambda \int_l K^*(x, s) q_i(s) d\tau(s),$$

where $q_i(t) \in \mathcal{L}_\tau^2$ is an arbitrary sequence of functions complete in \mathcal{L}_τ^2 . Let us note that the kernels $K(x, s)$ and $K^*(x, s)$ have a common spectrum of eigenvalues; hence it is easy to conclude that the sequence $p_i(t)$ will also be complete. Substituting these $p_i(t)$ into (6) and (7), we find

$$\psi_i(x) = \int_l H(x, t) q_i(t) d\tau(t), \quad (8)$$

$$b_{ij} = \int_l q_i(t) q_j(t) d\tau(t) - \lambda \int_a^b \psi_i(x) \psi_j(x) d\sigma(x). \quad (9)$$

Thus, for any real λ distinct from an eigenvalue, the following is valid.

Theorem 1. *The resolvent $\Gamma(x, s; \lambda)$ of the kernel $K(x, s)$, satisfying the conditions of the lemma and equality (1), is representable in the domain of definition of the kernel by a uniformly convergent series*

$$\Gamma(x, s; \lambda) = \sum_{i=1}^{\infty} \frac{1}{B_{i-1} B_i} \begin{vmatrix} \psi_1(x) & \dots & \psi_i(x) & & \\ b_{11} & \dots & b_{1i} & & \\ \dots & \dots & \dots & \dots & \\ b_{i-1,1} & \dots & b_{i-1,i} & & \end{vmatrix} \cdot \begin{vmatrix} \psi_1(s) & \dots & \psi_i(s) & & \\ b_{11} & \dots & b_{1i} & & \\ \dots & \dots & \dots & \dots & \\ b_{i-1,1} & \dots & b_{i-1,i} & & \end{vmatrix}, \quad (10)$$

where $B_n = |b_{ij}|_{i,j=1}^n$. The sequence of functions $q_i(t)$ from which, according to (8) and (9), the functions $\psi_i(x)$ and the numbers b_{ij} are constructed is complete in \mathcal{L}_σ^2 , and otherwise arbitrary.

We shall now show that if the loading function is assumed to be nondecreasing, then, after dropping the condition of positivity of the kernel and replacing the requirement of its continuity by the condition of continuity of the second iterated kernel, it is also possible to expand the resolvent in a series. In what follows we shall assume that the conditions listed here are fulfilled.

If $\Gamma(x, s; \lambda)$ is still the resolvent of the kernel $K(x, s)$ corresponding to the loading function $\sigma(x)$, then the function

$$G(x, s) = \int_a^b \Gamma(x, t; \mu) \Gamma(s, t; \bar{\mu}) d\sigma(t)$$

will, for an arbitrary complex μ , be a kernel satisfying the conditions of the lemma. (The preceding constructions are easily extended to kernels of this type.) It can be verified directly that, for $\mu = \frac{1}{2}(1 + i\sqrt{3})\lambda$ (λ real), the function

$$M(x, s; \lambda^2) = \frac{1}{\lambda} [\Gamma(x, s; \lambda) - K(x, s)]$$

will be the resolvent of the kernel $G(x, s)$ with loading function $\sigma(x)$.

On the basis of what has been said, we shall use series (10) to find the function $M(x, s; \lambda^2)$. Here, of course, λ^2 must be regarded as distinct from an eigenvalue of the kernel $G(x, s)$. This will be satisfied if λ is distinct from an eigenvalue of the kernel $K(x, s)$. Taking some sequence of functions $\eta_i(x)$ complete in \mathcal{L}_σ^2 and setting

$$q_i(x) = \eta_i(x) - \mu \int_a^b K(x, s) \eta_i(s) d\sigma(s),$$

we replace the functions $\psi_i(x)$ and the numbers b_{ij} entering into the expansion (10) of the resolvent $M(x, s; \lambda^2)$ respectively by the functions

$$\chi_i(x) = \int_a^b K(x, s) \eta_i(s) d\sigma(s) \tag{11}$$

and by the numbers

$$c_{ij} = \int_a^b \eta_i(x) \eta_j(x) d\sigma(x) - \lambda \int_a^b \chi_i(x) \eta_j(x) d\sigma(x). \tag{12}$$

It follows from this that

Theorem 2. *If the resolvent of a symmetric real kernel $K(x, s)$, whose second iterated kernel is continuous, is constructed*

with respect to a nondecreasing function $\sigma(x)$, then in the domain of definition of the kernel it is represented by the uniformly convergent series

$$\Gamma(x, s; \lambda) = K(x, s) + \lambda \sum_{i=1}^{\infty} \frac{1}{C_{i-1}C_i} \begin{vmatrix} \chi_1(x) \cdots \chi_i(x) \\ c_{11} \cdots c_{1i} \\ \cdots \cdots \cdots \\ c_{i-1,1} \cdots c_{i-1,i} \end{vmatrix} \cdot \begin{vmatrix} \chi_i(s) \cdots \chi_i(s) \\ c_{11} \cdots c_{1i} \\ \cdots \cdots \cdots \\ c_{i-1,1} \cdots c_{i-1,i} \end{vmatrix}, \quad (13)$$

where $C_n = |c_{ij}|_{i,j=1}^n$. The sequence of functions $\eta_i(x)$, according to which, by (11) and (12), the functions $\chi_i(x)$ and the numbers c_{ij} are constructed, is complete in L^2_σ , and otherwise arbitrary.

With the aid of the series (10) and (13) one can compute the resolvent approximately, replacing these series by their partial sums. In the case where the kernel satisfies the conditions of Theorem 1 and $\lambda < 0$, one can give a uniform estimate of the error of such an approximation.

Since for $\lambda < 0$ the determinants B_n and C_n are positive, the kernels $\Gamma(x, s; \lambda) - U_n(x, s; \lambda)$ and $V_n(x, s; \lambda) - \Gamma(x, s; \lambda)$ ($U_n(x, s; \lambda)$ and $V_n(x, s; \lambda)$ are, respectively, the partial sums of the series (10) and (13)) are positive. It follows that for an arbitrary function of bounded variation $Q(x)$ the inequality

$$\left[\int_a^b [\Gamma(x, t; \lambda) - U_n(x, t; \lambda)] dQ(t) \right]^2 \leq \int_a^b \int_a^b [V_n(s, t; \lambda) - V_n^*(s, t; \lambda)] dQ(s) dQ(t) [V_n(x, x; \lambda) - U_n(x, x; \lambda)].$$

Series of the form (10) and (13) lead to certain widely known methods for solving integral equations. Thus, for example, if in the series (10) the functions $q_i(t)$ are replaced by the functions $H(\xi_i, t)$ (ξ_i is a set of points dense in $[a, b]$), then one obtains the series to which Bateman's method² leads. The method of moments, or, as it is often called, Grammel's method³, also leads to the construction of the series (10). The solution proposed by Enskog⁴, when the orthonormality conditions of the functions considered by him are changed, coincides with the series (13). The authors of the methods listed do not prove the uniform convergence of the approximations they obtain, which makes it necessary to carry out the corresponding investigations in each particular problem. As is clear from

the proposed work, for the class of kernels considered such investigations are superfluous.

In conclusion we note that the series constructed are suitable for solving certain boundary-value differential problems. Lack of space does not permit us to touch on this question here.

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References Cited

¹ M. D. Dolberg, DAN, **120**, No. 5 (1958). ² H. Bateman, Proc. Roy. Soc. A, **100**, 441 (1922). ³ R. Grammel, Ing. Arch., **10**, 35 (1939). ⁴ D. Enskog, Dissert., Uppsala, 1917.

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