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Abstract

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ON THE EQUIVALENCE OF THE ROUTH-HURWITZ AND MARKOV STABILITY CRITERIA

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In the present work, proceeding from the works of P. L. Chebyshev and A. A. Markov on continued fractions ([1,2]) and their connection with the Routh–Hurwitz problem, proved by F. R. Gantmacher ([3]), the task was posed of finding a direct connection between the Hurwitz determinants and the Markov determinants. As a result this led to the formulation of a stability criterion equivalent to the Routh–Hurwitz criterion, called the Markov criterion, and also to the solution of the inverse stability problem for linear systems. Here the latter was interpreted as the selection of values of the coefficients of the characteristic equation satisfying the conditions of the required stability margin in terms of determinants.

§ 1. In order to reveal the connection between the Hurwitz determinants and the Markov determinants, let us consider the matrices $S^{(1)} = \{s_{iq}\}_1^n$, $A^{(1)} = \{a_{iq}\}_1^n$, and $H_n = \{h_{iq}\}_1^n$, where, in particular, H_n is the Hurwitz matrix composed of the coefficients $a_s = h_{iq}$ of the characteristic polynomial* $f(z)$.

Starting from $S^{(1)}$, we write:

$$\det\{s_{iq}\}_1^n = S^{(1)}\begin{pmatrix}12\dots n\\12\dots n\end{pmatrix} = \begin{vmatrix} 1 & s_{-1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & s_0 & 1 & s_{-1} & 0 & 0 & \cdots \\ 0 & -s_1 & 0 & s_0 & 1 & s_{-1} & \cdots \\ 0 & s_2 & 0 & -s_1 & 0 & s_0 & \cdots \\ 0 & -s_3 & 0 & s_2 & 0 & -s_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_1^n . \quad (1)$$

It can be proved for any n that the principal minors $S_s^* = S^{(1)}\binom{12\dots s}{12\dots s}$ $(s = 1, 2, \dots, n)$, generated by the matrix $S^{(1)}$, are both groups of Markov determinants (1):

$$S_1^* = 1; \qquad S_2^* = s_0; \qquad S_4^* = \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}; \qquad S_6^* = \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}; \; \dots$$

$$S_3^* = s_1; \qquad S_5^* = \begin{vmatrix} s_1 & s_2 \\ s_2 & s_3 \end{vmatrix}; \qquad S_7^* = \begin{vmatrix} s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \\ s_3 & s_4 & s_5 \end{vmatrix}; \; \dots \tag{2}$$

The lower triangular matrix $A^{(1)}$ is composed only of the odd coefficients of the characteristic equation, and

$$\det\{a_{iq}\}_1^n = A^{(1)}\begin{pmatrix} 12\dots n \\ 12\dots n \end{pmatrix} = \begin{vmatrix} a_1 & \cdot & \cdot & \dots \\ a_3 & a_1 & \cdot & \dots \\ a_5 & a_3 & a_1 & \dots \\ \cdot & \cdot & \cdot & \dots \end{vmatrix}_1^n . \tag{3}$$

The last element a_{iq} in (3) different from zero will be a_{2k+1}, if

* Real polynomials with constant coefficients are considered.

$n = 2k + 1$, and a_{2k-1}, if $n = 2k$. Taking into account that

$$\Delta_n = \det\{h_{iq}\}_1^n = H\begin{pmatrix} 12\dots n \\ 12\dots n \end{pmatrix} = \begin{vmatrix} a_1 & a_0 & 0 & \dots & \\ a_3 & a_2 & a_1 & \dots & \\ a_5 & a_4 & a_3 & \dots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_1^n , \tag{4}$$

where $h_{iq} = a_s \neq 0$, if $0 \leq s \leq n$, and $h_{iq} = a_s \equiv 0$, if $s < 0$ or $s > n$, we write

$$\Delta_n = H\begin{pmatrix} 12\dots n \\ 12\dots n \end{pmatrix} = A^{(1)}\begin{pmatrix} 12\dots n \\ 12\dots n \end{pmatrix} \cdot S^{(1)}\begin{pmatrix} 12\dots n \\ 12\dots n \end{pmatrix}. \tag{5}$$

By imposing the corresponding conditions on the values of the elements of the matrix $S^{(1)}$, one may require that, for any $n \geq 1$, the following determinant identities hold identically with respect to the index s:

$$\Delta_s = H\begin{pmatrix} 12\dots s \\ 12\dots s \end{pmatrix} \equiv A^{(1)}\begin{pmatrix} 12\dots s \\ 12\dots s \end{pmatrix} \cdot S^{(1)}\begin{pmatrix} 12\dots s \\ 12\dots s \end{pmatrix} \quad \text{for } s = 1, 2, \dots, n. \tag{6}$$

The equalities (6), taking into account the properties of the matrices $A^{(1)}$ and $S^{(1)}$, lead to the following dependence between the Hurwitz determinants Δ_s and the Markov determinants S_s^*:

$$\Delta_s = a_1^s S_s^* \qquad (s = 1, 2, \dots, n). \tag{7}$$

At the same time, the order of the determinants S_s^* is considerably lower than the order of Δ_s. (For the computation of Markov determinants of higher order one may apply the convenient "Krakovian method" ([7],[8]).)

The conditions necessary and sufficient for uniquely determining all elements s_β of the matrix $S^{(1)}$ through the coefficients of the characteristic polynomial are obtained as a result of multiplying the determinants on the right-hand side of equality (5) according to the formula

$$h_{iq} = \sum_{\alpha=1}^{n} a_{i\alpha} s_{\alpha q}. \tag{8}$$

They constitute n linear equations, which we shall call the independent correspondence conditions (between the coefficients of the characteristic polynomial and the Markov parameters s_β). It is expedient to divide them into two groups. The first of them corresponds to the significant elements of the matrix H_n, and the second to the zero elements of this matrix.

$$\varkappa = k+1 \begin{cases} a_0 = a_1 s_{-1}, \\ a_2 = a_3 s_{-1} + a_1 s_0, \\ a_4 = a_5 s_{-1} + a_3 s_0 - a_1 s_1, \\ \dots \\ a_{2k} = a_{2k+1} s_{-1} + a_{2k-1} s_0 - a_{2k-3} s_1 + \dots + (-1)^{k-1} a_1 s_{k-1} \end{cases}, \tag{9}$$

where $k = E(n/2)$ and $a_{2k+1} \equiv 0$, if $n = 2k$;

$$0 = \sum_{i=0}^{n-k-1} (-1)^{(q-i)} a_{1+2i} s_{q-i} \qquad (q = k, k+1, \dots, n-2). \tag{10}$$

Equations (9), whose number is $\alpha = k + 1$, represent the first group, and equations (10), whose number is $n - \alpha$, the second group of independent correspondence conditions. We illustrate the joint notation of expressions (9) and (10) by the example $n = 2k + 1 = 5$:

$$\alpha = k+1 \begin{cases} a_0 = a_1 s_{-1}, \\ a_2 = a_3 s_{-1} + a_1 s_0, \\ a_4 = a_5 s_{-1} + a_3 s_0 - a_1 s_1; \end{cases}$$

$$n - \alpha \begin{cases} 0 = a_5 s_0 - a_3 s_1 + a_1 s_2, \\ 0 = -a_5 s_1 + a_3 s_2 - a_1 s_3. \end{cases}$$

On the basis of the results obtained, in particular the dependences (7) and the relations (15)*, given below, we formulate stability criteria equivalent to the Routh–Hurwitz and Liénard–Chipart criteria ([3],[4],[6]). Their foundations were laid by A. A. Markov in his work "On functions obtained when converting series into continued fractions" ([1]), published in 1894; we shall therefore call them the Markov and Liénard–Chipart–Markov criteria. In doing so we shall proceed from the necessary stability conditions for linear systems established by A. Stodola, consisting in the constancy of the signs of all coefficients of the polynomial $f(z)$.

Markov stability criterion. *In order that all roots of the characteristic real polynomial* $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ *have negative real parts, it is necessary and sufficient that all its coefficients be of one sign and that all the corresponding Markov determinants*** *be positive.*

Liénard–Chipart–Markov stability criterion. *In order that all roots of the characteristic real polynomial* $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ *have negative real parts, it is necessary and sufficient that all its coefficients be of one sign and that only the corresponding even or only the odd Markov determinants be positive.*

A consequence of the latter is the lemma:

Lemma. *The determinantal inequalities* $S_s^* > 0$ *or* $\overline{S}_s^* > 0$ $(s = 1, 2, \dots, n)$, *where* S_s^* *and* $\overline{S}_s^*$ *correspond to a characteristic real polynomial satisfying the Stodola conditions, are not independent, since from the positivity of the even Markov determinants follows the positivity of the odd ones, and conversely.*

§ 2. The determinants S_s^* are expressed linearly in terms of the senior Markov parameters****. Consequently, starting from the prescribed values of the Hurwitz determinants Δ_s and the dependences (7), one can find the values of all parameters s_β. This makes it possible to determine uniquely the values of all coefficients of the characteristic equation. In doing so it is necessary to consider separately the cases of odd and even degree n.

Case $n = 2k + 1$. The odd coefficients of the characteristic equation are determined by the formula

$$a_{2i+1} = (-1)^{i+1} a_1 \frac{C_q^{(0)} \begin{pmatrix} 12 \dots \dots k \\ 12 \dots q \dots k \end{pmatrix}}{S_k^{(0)} \begin{pmatrix} 12 \dots k \\ 12 \dots k \end{pmatrix}}; \quad i = 1, 2, \dots, k = \mathrm{E}\left(\frac{n}{2}\right); \; q = 1 + k - i, \tag{11}$$

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* The dependences (7), as well as the dependences (15), in contrast to the

dependence known in the literature ([3]-[5])

$$R_p = a_0^{2p} \, |s_{k+l}|_0^{p-1} = \nabla_{2p},$$

determine a direct and the most general connection between the Hurwitz and Markov determinants, valid for both even and odd degrees of the characteristic equation.

** The formulated criterion is obtained from the theorems of A. A. Markov on determinants and roots (([1]), p. 96). It represents a generalization of Theorem 17 of F. R. Gantmakher from (([3]), p. 468) to all both even and odd degrees of the characteristic equation under a single law for forming the Markov parameters.

*** In the case of the initial dependence (7) these will be the determinants S_s^*, and in the case of dependence (15) the determinants $\overline{S}_s^*$. The determinants S_s^* and $\overline{S}_s^*$ differ essentially from one another in the manner of forming their elements (the Markov parameters).

It should be noted that in the case of the determinants $\overline{S}_s^*$ one may in general disregard the Stodola conditions, and in the case of S_s^* replace these conditions only by $\operatorname{sign} a_0 a_1 > 0$. However, the Stodola conditions have been included in the formulation of the criterion in connection with the fact that their nonfulfillment is the first indication of instability.

**** A senior Markov parameter is the element s_β of the determinant S_s^* with the largest value of the index β.

where

$$S_k^{(0)} = \begin{vmatrix} s_0 & s_1 & \dots & s_{k-1} \\ s_1 & s_2 & \dots & s_k \\ . & . & . & . \\ s_{k-1} & s_k & \dots & s_{2k-2} \end{vmatrix}; \qquad C^{(0)} = \begin{vmatrix} s_0 & s_1 & \dots & s_{k-1} & s_k \\ s_1 & s_2 & \dots & s_k & s_{k+1} \\ . & . & . & . & . \\ s_{k-1} & s_k & \dots & s_{2k-2} & s_{2k-1} \end{vmatrix} \quad (12)$$

and $C_q^{(0)}\binom{12......k}{12...q...k}$ is a determinant of order k, which is obtained from the augmented matrix $C^{(0)}$ by transposing its q-th column to the last, i.e., the $(k+1)$-st. After the values of all odd coefficients a_{2i+1} have been found, it is easy, with the aid of the first group of $\alpha = k+1$ independent correspondence conditions, to determine all even coefficients a_{2i}.

Case $n = 2k$**.** The odd coefficients are found by the formula

$$a_{2i+1} = (-1)^{i+1} a_1 \frac{C_q^{(1)}\binom{12......k-1}{12...q...k-1}}{S_{k-1}^{(1)}\binom{12...k-1}{12...k-1}}, \qquad i = 1, 2, \dots, k-1; \quad q = k - i, \quad (13)$$

where

$$S_{k-1}^{(1)} = \begin{vmatrix} s_1 & s_2 & \dots & s_{k-1} \\ s_2 & s_3 & \dots & s_k \\ \cdot & \cdot & \cdot & \cdot \\ s_{k-1} & s_k & \dots & s_{2k-3} \end{vmatrix}; \qquad C^{(1)} = \begin{vmatrix} s_1 & s_2 & \dots & s_{k-1} & s_k \\ s_2 & s_3 & \dots & s_k & s_{k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{k-1} & s_k & \dots & s_{2k-3} & s_{2k-2} \end{vmatrix} \quad (14)$$

and $C_q^{(1)}\binom{12\dots\dots,k-1}{12\dots q\dots,k-1}$ is a determinant of order $(k-1)$, which is obtained from the augmented matrix $C^{(1)}$ by replacing its q-th column by the last, i.e., the k-th. The even coefficients are determined, analogously to the preceding case, with the aid of the first group of independent correspondence conditions.

It can also be proved that the relation

$$\Delta_s = a_0^s \bar{S}_s^* \qquad (s = 1, 2, \dots, n), \quad (15)$$

holds, which is obtained as a result of solving the determinant equality

$$\Delta_n = H\binom{12\dots n}{12\dots n} \equiv A^{(0)}\binom{12\dots n}{12\dots n} \cdot S^{(0)}\binom{12\dots n}{12\dots n}, \quad (16)$$

where

$$A^{(0)} = \begin{vmatrix} a_0 & \cdot & \cdot & \dots \\ a_2 & a_0 & \cdot & \dots \\ a_4 & a_2 & a_0 & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_1^n; \qquad S^{(0)} = \begin{vmatrix} \bar{s}_0 & 1 & 0 & 0 & \dots \\ -\bar{s}_1 & 0 & \bar{s}_0 & 1 & \dots \\ \bar{s}_2 & 0 & -\bar{s}_1 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_1^n;$$

$$\bar{S}_1^* = \bar{s}_0; \quad \bar{S}_2^* = \bar{s}_1; \quad \bar{S}_3^* = \begin{vmatrix} \bar{s}_0 & \bar{s}_1 \\ \bar{s}_1 & \bar{s}_2 \end{vmatrix}; \quad \bar{S}_4^* = \begin{vmatrix} \bar{s}_1 & \bar{s}_2 \\ \bar{s}_2 & \bar{s}_3 \end{vmatrix}; \quad \bar{S}_5^* = \begin{vmatrix} \bar{s}_0 & \bar{s}_1 & \bar{s}_2 \\ \bar{s}_1 & \bar{s}_2 & \bar{s}_3 \\ \bar{s}_2 & \bar{s}_3 & \bar{s}_4 \end{vmatrix}; \dots$$

In practical applications, relations (7) are more convenient than (15).

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Note: Figure translations are in progress. See original paper for figures.