Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1960. Vol. 132, No. 2

MATHEMATICS

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APPROXIMATION OF FUNCTIONS WITH A GIVEN MODULUS OF SMOOTHNESS ON THE EXTERIOR OF AN INTERVAL AND A HALF-LINE

(Presented by Academician V. I. Smirnov on January 15, 1960)

A. F. Timan proved the following theorem (^{7}):

If a function $f(x)$, defined on $[-1, 1]$, has there a continuous r-th derivative, then for every natural number n there exists an ordinary polynomial $P_n(x)$ of degree not exceeding n, satisfying, for every $x \in [-1, 1]$, the inequality

$$|f(x) - P_n(x)| \le C_r \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^r \omega^{(r)}\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right),$$

where

$$\omega^{(r)}(h) = \omega(f^{(r)}; h) = \sup_{|x_1 - x_2| \le h} |f^{(r)}(x_1) - f^{(r)}(x_2)|, \quad x_1, x_2 \in [-1, 1],$$

is the modulus of continuity of the r-th derivative; C_r does not depend on x or n.

This theorem (unlike the well-known Jackson theorem for a finite interval) admits an inversion to the same extent as Jackson' s theorem in the periodic case: the inverse theorems to it (^{8}); the case $\omega^{(r)}(t) = t^\alpha$ $(0 < \alpha < 1)$, see (^{4}), make it possible to judge structural properties of functions on the whole interval, and for some moduli of continuity give a complete inversion of this theorem. V. K. Dzyadyk (^{5}) generalized the indicated theorem, replacing the modulus of continuity by the modulus of smoothness*

$$\omega_2(h) = \sup_{|x_1 - x_2| \le 2h} \left| f(x_1) - 2f\left(\frac{x_1 + x_2}{2}\right) + f(x_2)\right|, \quad x_1, x_2 \in [-1, 1].$$

Yu. A. Brudnyi (3) obtained theorems similar to A. F. Timan' s theorem in the case of approximation by entire functions on the exterior of an interval and on a half-line. The inverse theorems to them also belong to him (3).

In the present work the indicated results of Yu. A. Brudnyi are generalized. Let the function $f(x)$ have on

$$E = (-\infty, -1] \cup [1, \infty)$$

r uniformly continuous and bounded derivatives, and let

$$\omega_2^{(r)}(h) = \omega_2(f^{(r)}; h)$$

be the modulus of smoothness of the r-th derivative, while B_σ is the class of entire functions of degree not exceeding σ, bounded on the real axis. Then the following holds:

Theorem 1. For every $\sigma \geq 1$ there exists an entire function $G_\sigma(f; x) \in B_\sigma$ such that, for every $x \in E$,

$$|f(x) - G_\sigma(f; x)| \leq C_r \left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2} \right)^r \omega_2^{(r)} \left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2} \right),$$

where C_r does not depend on x or σ.

The proof of the theorem is based on the following lemmas.

Lemma 1. If $\omega_2(h) \neq 0$ for $h \neq 0$, then for $h \leq 1$

$$\omega^2(h) \leq C\omega_2(h); \tag{1}$$

$$\omega(h^2) \leq C\omega_2(h), \tag{2}$$

where C does not depend on h;

$$\omega(h) = \sup_{\substack{x \\ 0 \leq \delta \leq h}} |f(x+\delta) - f(x)|.$$

Lemma 2. If $f(x)$ and $g(x)$ are bounded and $\omega_2(f; h) \neq 0$ for $h \neq 0$, then

$$\omega_2(fg; h) \leq C[\omega_2(f; h) + \omega_2(g; h)].$$

where C does not depend on h.

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* The same result was obtained by Freud (9).

Lemma 3. A function $f(x)$ bounded on E can be extended to the whole axis with preservation (up to a constant) of $\omega_2(h)$ and $\omega(h)$. The exception is formed by the functions

$$f_{a,b}(x) = a, \quad x \geq 1; \qquad f_{a,b} = b, \quad x \leq -1$$

when $a \neq b$.

Lemma 2 is proved with the aid of inequality (1), and Lemma 3 with the aid of Lemma 2.

Let us prove Lemma 1. From the equality

$$2[f(x+\delta) - f(x)] = [f(x-\delta) - 2f(x) + f(x+\delta)] + [f(x+\delta) - f(x-\delta)]$$

it follows that $\omega(h) \leq \frac{1}{2}\omega_2(h) + \frac{1}{2}\omega(2h)$, whence for any natural k

$$\omega(h) \leq \sum_{\nu=0}^{k} \frac{\omega_2(2^\nu h)}{2^{\nu+1}} + \frac{\omega(2^{k+1}h)}{2^{k+1}}. \tag{3}$$

Taking into account that $\omega_2(2^\nu h) \leq 2^{2\nu}\omega_2(h)$, and putting

$$k = \left[\log_2 \frac{1}{\sqrt{\omega_2(h)}}\right]$$

(one may assume that $\omega_2(1) = 1$), we obtain from (3) inequality (1), and putting

$$k = \left[\log_2 \frac{1}{\sqrt{h}}\right]$$

and using the monotonicity of $\omega_2(h)$ and inequality (1), we obtain from (3) inequality (2).

Inequality (2) also follows from Marchaud' s inequality (^{6}):

$$\omega_k(h) = \sup_{\substack{x \\ 0 \leq \delta \leq h}} \left| \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} f(x+\nu\delta) \right| \leq C_k h^k \int_h^1 \frac{\omega_{k+1}(u)}{u^{k+1}}\, du \quad \text{for } h \leq \frac{1}{2}.$$

In proving the theorem kernels $K(u)$ with the following properties are used: $K(u)$ is an entire function of degree not exceeding one; $K(u) = K_p(u) = K(-u)$, $(|u|+1)^p|K(u)| = O(1)$ (p an integer ≥ 2); $\int_{-\infty}^{\infty} K(u)\, du = 1^*$.

These conditions are satisfied, for example, by kernels of Jackson type

$$K(u) = \frac{1}{\gamma_p}\left(\frac{\sin\frac{u}{p}}{u}\right)^p, \qquad \gamma_p = \int_{-\infty}^{\infty}\left(\frac{\sin\frac{u}{p}}{u}\right)^p du.$$

The role of the kernels $K(u)$ in approximation questions consists in the fact that the expression

$$\int_{-\infty}^{\infty} f\left(x+\frac{t}{\sigma}\right) K(t)\, dt$$

is an entire function of degree not exceeding σ, even in the case of evenness of $f(x)$.

Let us now prove the theorem for $r=0$. It is possible to consider only even and odd functions $f(x)$. Let $f(x)=f(-x)$. Consider the single-valued function

$$\tilde{f}(u)=f\left(\sqrt{1+u^2}\right).$$

We shall show that the desired function $G_\sigma(f;x)$ is the expression

$$g_\sigma(\tilde{f};u)=2\int_{-\infty}^{\infty}\tilde{f}\left(u+\frac{t}{\sigma}\right)K(t)\,dt-\int_{-\infty}^{\infty}\tilde{f}\left(u+\frac{\sqrt{2}\,t}{\sigma}\right)K(t)\,dt=\sum_{\nu=0}^{\infty}C_\nu u^{2\nu},$$

if instead of u we substitute $\sqrt{x^2-1}$. Extend $f(x)$ by Lemma 3. Taking into account that

$$\tilde{f}\left(u+\frac{t}{\sigma}\right)=f\left(T+A\frac{t}{\sigma}+B\frac{t^2}{\sigma^2}+C\frac{t^3}{\sigma^3}+\widetilde{D}\frac{t^4}{\sigma^4}\right),$$

where

$$T=\sqrt{1+u^2},\qquad A=\frac{u}{\sqrt{1+u^2}},\qquad B=\frac{1}{2(1+u^2)^{3/2}},\qquad C=-\frac{1}{2(1+u^2)^{5/2}},$$

$|\widetilde{D}|<1/2$, and also that

* $K(u)$ is a special case of kernels of Fejér type (see (^{1}), p. 126).

$$\begin{aligned}
&\left| f(T) - f\left(T + A\frac{t}{\sigma} + B\frac{t^2}{\sigma^2} + C\frac{t^3}{\sigma^3}\right) - f\left(T - A\frac{t}{\sigma} + B\frac{t^2}{\sigma^2} - C\frac{t^3}{\sigma^3}\right)\right.\\
&\quad\left. + \frac{1}{2} f\left(T + A\frac{\sqrt{2}t}{\sigma} + B\frac{2t^2}{\sigma^2} + C\frac{2\sqrt{2}t^3}{\sigma^3}\right) + \frac{1}{2} f\left(T - A\frac{\sqrt{2}t}{\sigma} + B\frac{2t^2}{\sigma^2} - C\frac{2\sqrt{2}t^3}{\sigma^3}\right)\right|\\
&\leq \left| 2f\left(T + B\frac{t^2}{\sigma^2}\right) - f\left(T + A\frac{t}{\sigma} + B\frac{t^2}{\sigma^2} + C\frac{t^3}{\sigma^3}\right) - f\left(T - A\frac{t}{\sigma} + B\frac{t^2}{\sigma^2} - C\frac{t^3}{\sigma^3}\right)\right|\\
&\quad + \frac{1}{2}\left| f\left(T + A\frac{\sqrt{2}t}{\sigma} + B\frac{2t^2}{\sigma^2} + C\frac{2\sqrt{2}t^3}{\sigma^3}\right)\right.\\
&\qquad\left. + f\left(T - A\frac{\sqrt{2}t}{\sigma} + B\frac{2t^2}{\sigma^2} - C\frac{2\sqrt{2}t^3}{\sigma^3}\right) - 2f\left(T + B\frac{2t^2}{\sigma^2}\right)\right|\\
&\quad + \left| f\left(T + B\frac{2t^2}{\sigma^2}\right) - 2f\left(T + B\frac{t^2}{\sigma^2}\right) + f(T)\right|\\
&\leq C_1 \omega_2\left(\frac{|At|}{\sigma} + \frac{|Bt^2|}{\sigma^2} + \frac{|Ct^3|}{\sigma^3}\right),
\end{aligned}$$

where C_1 does not depend on t and σ, we obtain

$$|f(x) - G_\sigma(f; x)| = |\tilde{f}(u) - g_\sigma(\tilde{f}; u)| \leq$$

$$\leq C_1 \int_{-\infty}^{\infty} \omega_2\left(\frac{|At|}{\sigma} + \frac{|Bt^2|}{\sigma^2} + \frac{|Ct^3|}{\sigma^3}\right) |K(t)|\, dt + C_2 \int_{-\infty}^{\infty} \omega\left(\frac{t^4}{\sigma^4}\right) |K(t)|\, dt.$$

It remains to take into account that, for $\lambda > 0$, $\omega_2(\lambda h) \leq (\lambda + 1)^2 \omega_2(h)$, to apply Lemma 1, and to take $K(u) = K_8(u)$.

If $f(x) = -f(-x)$, we apply what has been proved to the function

$$F(x) = \frac{f(x)}{g_1(x)} \qquad \left(g_1(x) = \int_0^x \frac{\sin^2 \frac{t}{2}}{t^2}\, dt\right),$$

taking into account that $g_1(x)$ is an entire function of degree one, and $\omega_2(F) < C\omega_2(f; h)$ (Lemma 2).

Suppose that the theorem has been proved in the case of existence of the $(r-1)$-st derivative, and first let $f(x) = f(-x)$. Then there exists an even entire function $G_\sigma(x) \in B_\sigma$ such that

$$|f'(x) - G_\sigma(x)| \le C_{r-1}\left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2}\right)^{r-1} \omega_2^{(r)}\left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2}\right).$$

Let

$$R(x) = \int_0^x [f'(t) - G_\sigma(t)]\,dt \quad \text{for } x \ge 1, \qquad R(x) = R(-x).$$

By means of Lagrange' s mean value theorem we obtain

$$\left|R\left(\sqrt{1+u^2}\right) - \int_{-\infty}^{\infty} R\left(\sqrt{1+\left[u+\frac{t}{\sigma}\right]^2}\right) K(t)\,dt\right| \le$$

$$\le C_{r-1}C_3\left(\frac{|u|}{\sqrt{1+u^2}}\frac{1}{\sigma} + \frac{1}{\sigma^2}\right)^r \omega_2^{(r)}\left(\frac{|u|}{\sqrt{1+u^2}}\frac{1}{\sigma} + \frac{1}{\sigma^2}\right)\int_{-\infty}^{\infty}(t^2+1)|K(t)|\,dt.$$

It remains to take $K(u) = K_4(u)$ and instead of u substitute $\sqrt{x^2-1}$. The proof is completed as in the case $r = 0$. Theorem 1 is proved.

The following theorem, converse to Theorem 1, holds:

Theorem 2. Let, for a function $f(x)$ given on E, for some $r \ge 0$ and a function $\tilde{\omega}_2(h) \ne 0$, given on $[0, 2]$ and satisfyi-

satisfying there the inequality $\tilde{\omega}_2(\lambda h) \le (\lambda+1)^2\tilde{\omega}_2(h)$ and such that

$$\int_0^1 \nu(r)\frac{\tilde{\omega}_2(u)}{u^3}\,du < \infty$$

($\nu(r) = 0$ for $r = 0$; $\nu(r) = 1$ for $r \ge 1$), for any $\sigma \ge 1$ there is an entire function $G_\sigma(x) \in B_\sigma$ such that, for all $x \in E$,

$$|f(x) - G_\sigma(x)| \le \left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2}\right)^r \tilde{\omega}_2\left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2}\right).$$

Then $f(x)$ has on E r uniformly continuous and bounded derivatives, and for $h \le 1$

$$\omega_2\left(f^{(r)}; h\right) \le C\left\{h^2\int_h^2 \frac{\tilde{\omega}_2(u)}{u^3}\,du + \int_0^h \nu(r)\frac{\tilde{\omega}_2(u)}{u}\,du\right\},$$

where C does not depend on h.

The proof is carried out by the known method of S. N. Bernstein and is based on the following inequality of Yu. A. Brudnyi (^{3}):

If, for some nonnegative r, for all $x \in E$

$$|G_\sigma(x)| \leq \left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2}\right)^r \omega\left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2}\right),$$

then

$$|G'_\sigma(x)| \leq C\left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2}\right)^{r-1} \omega\left(\frac{\sqrt{x^2-1}}{|x|\sigma} + \frac{1}{\sigma^2}\right),$$

where $\omega(h)$ is any modulus of continuity. From the proof of this inequality it is clear that it remains valid when $\omega(h)$ is replaced by $\tilde{\omega}_2(h)$, and also for negative r.

Corollary. If the function $f(x)$ satisfies the conditions of Theorem 2 and

$$h^2 \int_h^2 \frac{\tilde{\omega}_2(u)}{u^3}\, du + \int_0^h \nu(r) \frac{\tilde{\omega}_2(u)}{u}\, du = O\left[\tilde{\omega}_2(h)\right],$$

then

$$\omega_2\left(f^{(r)}; h\right) = O\left[\tilde{\omega}_2(h)\right].$$

Condition (4) is satisfied, for example, by the function $\tilde{\omega}_2(h) = h$. If the function $f(x)$ is given on $[0, \infty)$, then, as S. N. Bernstein showed ((^{2}), see also (^{3})), the natural apparatus of approximation is provided by entire functions of finite half-degree σ, i.e. such entire functions $H_\sigma(u)$ that $G_\sigma(u) = H_\sigma(u^2)$ are entire functions of finite degree not exceeding σ.

Analogously to Theorem 1, the following theorem is proved.

Theorem 3. If $f(x)$ has on $[0, \infty)$ r uniformly continuous and bounded derivatives, then for any $\sigma \geq 1$ there exists an entire function $H_\sigma(x)$ of finite half-degree σ such that, for all $x \in [0, \infty)$,

$$|f(x) - H_\sigma(x)| \leq C_r \left(\frac{\sqrt{x}}{\sigma} + \frac{1}{\sigma^2}\right)^r \omega_2^{(r)}\left(\frac{\sqrt{x}}{\sigma} + \frac{1}{\sigma^2}\right),$$

where $\omega_2^{(r)}(h)$ is the modulus of smoothness of the r-th derivative, and C_r does not depend on x and σ.

There is a converse theorem analogous to Theorem 2. We note that the proof of Theorem 1 given above is also applicable in the case of approximation by algebraic polynomials on a finite interval.

In conclusion, I express my gratitude to Prof. A. F. Timan for posing the problem and for his guidance.

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named after the 300th anniversary of the reunification of Ukraine with Russia

Received
13 I 1960

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Note: Figure translations are in progress. See original paper for figures.

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