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Abstract

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ON A NEW FORM OF DISPERSION RELATIONS

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The dispersion relations usually used between the imaginary part $\operatorname{Im} f(E)$ and the real part $\operatorname{Re} f(E)$ of the scattering amplitude $f(E)$, analytic in the upper half-plane $\operatorname{Im} E > 0$, are obtained on the basis of using Cauchy' s theorem as a criterion for the analyticity of $f(E)$ (see, for example, ([1])).

In the present paper new dispersion relations are obtained, based on using the M. V. Keldysh—L. I. Sedov theorem ([2]) as a criterion for the analyticity of $f(E)$. At the same time, new additional relations are also obtained, which the real and imaginary parts of the scattering amplitude must necessarily satisfy.

1. For simplicity in formulating the results obtained, let us consider the case when $f(E)$ is a forward scattering amplitude that decreases sufficiently rapidly as $|E| \to \infty$. We give here the formulation of the Keldysh—Sedov theorem ([2]) needed below.

Let the real axis $E \in (-\infty, \infty)$ be divided by the points $a_1, b_1, \ldots, \ldots, a_m, b_m$ ($a_1 < b_1 < a_2 < \cdots < b_m$) into $2m$ intervals, and let on the intervals $E \in (a_k, b_k)$ ($k = 1, \ldots, m$) the real part $\operatorname{Re} f(E)$ be given, and on the intervals $E \in (b_k, a_{k+1})$ ($k = 1, \ldots, m$; $a_{m+1} \equiv a_1$) the imaginary part $\operatorname{Im} f(E)$ of a function $f(E)$, analytic in the upper half-plane $\operatorname{Im} E > 0$, which as $|E| \to \infty$ tends to zero sufficiently rapidly and is bounded at all division points a_k, b_k. Then this analytic function $f(E)$ is determined in the following way by its real and imaginary parts on the intervals (a_k, b_k), (b_k, a_{k+1}), respectively:

$$f(E) = \frac{R(E)}{i\pi} \sum_{k=1}^{m} \left\{ \int_{a_k}^{b_k} \frac{\operatorname{Re} f(E')}{R(E')} \frac{dE'}{E' - E} + i \int_{b_k}^{a_{k+1}} \frac{\operatorname{Im} f(E')}{R(E')} \frac{dE'}{E' - E} \right\}, \quad (1)$$

where $R(E)$ is a single-valued branch of the function

$$R(E) \equiv \left[\prod_{k=1}^{m} (E - a_k)(E - b_k) \right]^{1/2},$$

and, because $f(E)$ is finite at the division points a_k, b_k, the real part $\operatorname{Re} f(E)$ and the imaginary part $\operatorname{Im} f(E)$ must necessarily satisfy the additional conditions:

$$\sum_{k=1}^{m}\left\{\int_{a_k}^{b_k}\frac{\operatorname{Re} f(E)}{R(E)}\,E^{l-1}dE+i\int_{b_k}^{a_{k+1}}\frac{\operatorname{Im} f(E)}{R(E)}\,E^{l-1}dE\right\}=0 \qquad (l=1,2,\dots,m). \tag{2}$$

If, however, on the intervals $E\in(a_k,b_k)$ the imaginary part $\operatorname{Im} f(E)$ is given, and on the intervals $E\in(b_k,a_{k+1})$ the real part $\operatorname{Re} f(E)$, then (1) becomes

$$f(E)=-\frac{R(E)}{\pi}\sum_{k=1}^{m}\left\{-\int_{a_k}^{b_k}\frac{\operatorname{Im} f(E')}{R(E')}\,\frac{dE'}{E'-E}+i\int_{b_k}^{a_{k+1}}\frac{\operatorname{Re} f(E')}{R(E')}\,\frac{dE'}{E'-E}\right\}, \tag{3}$$

and the additional conditions (2) become

$$\sum_{k=1}^{m}\left\{-\int_{a_k}^{b_k}\frac{\operatorname{Im} f(E)}{R(E)}E^{l-1}\,dE+i\int_{b_k}^{a_{k+1}}\frac{\operatorname{Re} f(E)}{R(E)}E^{l-1}\,dE\right\}=0 \quad (l=1,2,\dots,m). \tag{4}$$

2. Let us write out in detail, on the basis of the Keldysh–Sedov formulas (1), (3), the new dispersion relations and additional conditions, based on (2), (4), for two important special cases.

Let $m=1$ and $a_1=-b_1=-E_1$ $(E_1>0)$. Then from (3) we obtain, as is easy to see, the following dispersion relation for $E>E_1$*:

$$\operatorname{Im} f(E)=-\frac{R(E;E_1)}{\pi}\left\{P\int_{E_1}^{\infty}\frac{\operatorname{Re} f(E')}{R(E';E_1)}\frac{dE'}{E'-E}+P\int_{-\infty}^{-E_1}\frac{\operatorname{Re} f(E')}{R(E';E_1)}\frac{dE'}{E'-E}+\int_{-E_1}^{E_1}\frac{\operatorname{Im} f(E')}{R(E';E_1)}\frac{dE'}{E'-E}\right\}, \tag{5}$$

and the additional conditions (2) and (4) are written in the form

$$\int_{-E_1}^{E_1}\frac{\operatorname{Re} f(E)}{R(E;E_1)}\,dE=\int_{E_1}^{\infty}\frac{\operatorname{Im} f(E)}{R(E;E_1)}\,dE-\int_{-\infty}^{-E_1}\frac{\operatorname{Im} f(E)}{R(E;E_1)}\,dE; \tag{6}$$

$$\int_{-E_1}^{E_1}\frac{\operatorname{Im} f(E)}{R(E;E_1)}\,dE=\int_{-\infty}^{-E_1}\frac{\operatorname{Re} f(E)}{R(E;E_1)}\,dE-\int_{E_1}^{\infty}\frac{\operatorname{Re} f(E)}{R(E;E_1)}\,dE. \tag{7}$$

Let now $m=2$ and $a_1=-b_2=-E_2$ $(E_2>0)$, $b_1=-a_2=-E_1$ $(E_2>E_1>0)$. Then it is not difficult to obtain from (3) the following dispersion relation for $E_2>E>E_1$:

$$\operatorname{Im} f(E) = \frac{R(E;E_1)R(E;E_2)}{\pi}\left\{\int_{-\infty}^{-E_2} \frac{\operatorname{Im} f(E')}{R(E';E_1)R(E';E_2)}\frac{dE'}{E'-E}\right.$$

$$+P\int_{-E_2}^{-E_1} \frac{\operatorname{Re} f(E')}{R(E';E_1)R(E';E_2)}\frac{dE'}{E'-E} - \int_{-E_1}^{E_1} \frac{\operatorname{Im} f(E')}{R(E';E_1)R(E';E_2)}\frac{dE'}{E'-E}$$

$$\left.-P\int_{E_1}^{E_2} \frac{\operatorname{Re} f(E')}{R(E';E_1)R(E';E_2)}\frac{dE'}{E'-E} + \int_{E_2}^{\infty} \frac{\operatorname{Im} f(E')}{R(E';E_1)R(E';E_2)}\frac{dE'}{E'-E}\right\}. \tag{8}$$

The additional conditions (2), (4) become, respectively,

$$\int_{-\infty}^{-E_2} \frac{\operatorname{Re} f(E)}{R(E;E_1)R(E;E_2)}\,dE - \int_{-E_1}^{E_1} \frac{\operatorname{Re} f(E)}{R(E;E_1)R(E;E_2)}\,dE + \int_{E_2}^{\infty} \frac{\operatorname{Re} f(E)}{R(E;E_1)R(E;E_2)}\,dE$$

$$= \int_{-E_2}^{-E_1} \frac{\operatorname{Im} f(E)}{R(E;E_1)R(E;E_2)}\,dE - \int_{E_1}^{E_2} \frac{\operatorname{Im} f(E)}{R(E;E_1)R(E;E_2)}\,dE; \tag{9}$$

$$\int_{-E_1}^{E_1} \frac{\operatorname{Im} f(E)}{R(E;E_1)R(E;E_2)}\,dE - \int_{-\infty}^{-E_2} \frac{\operatorname{Im} f(E)}{R(E;E_1)R(E;E_2)}\,dE - \int_{E_2}^{\infty} \frac{\operatorname{Im} f(E)}{R(E;E_1)R(E;E_2)}\,dE$$

$$= \int_{-E_2}^{-E_1} \frac{\operatorname{Re} f(E)}{R(E;E_1)R(E;E_2)}\,dE - \int_{E_1}^{E_2} \frac{\operatorname{Re} f(E)}{R(E;E_1)R(E;E_2)}\,dE. \tag{10}$$

* After the present work had been reported and prepared for press, we learned that a dispersion relation analogous to (5) had earlier been proposed in the work of V. Gilbert [3]; however, in that work the necessary additional conditions (6), (7) were not obtained.

The two remaining additional conditions are obtained from (9) and (10) by multiplying the integrands by E. In (5)–(10) the notation $R(E;E_{1,2}) \equiv \sqrt{E^2 - E_{1,2}^2}$ has been introduced. Analogous dispersion relations and additional conditions can also be written down for other values $m = 3, 4, \ldots$.

3. Let us proceed to a discussion of the dispersion relations (5), (8) obtained and of the additional conditions (6), (7), (9), (10). We first note that, on the basis of the symmetry condition (1),

$$f^*(-E) = f_{\text{an}}(E), \tag{11}$$

where $f_{\text{an}}(E)$ is the scattering amplitude for antiparticles. Using the symmetry condition (11), one can in the usual way reduce all integrals in (5)–(10) to integrals over positive energies. For physical reasons it is natural to choose $E_1 = \mu$, where μ is the rest mass of the scattered particles, so that $[-E_1, E_1]$ corresponds to the "unphysical" region of energies E.

By themselves, the dispersion relations (5), (8) in principle carry no new (independent) information in comparison with the ordinary dispersion relations ([1]), since both the former and the latter are derived on the basis of one and the same fact—the analyticity of the forward scattering amplitude $f(E)$ in the upper half-plane $\operatorname{Im} E > 0$ ([1]). Practically, however, taking into account that the experimental data $(\operatorname{Im} f(E), \operatorname{Re} f(E))$ are determined with unavoidable errors (noise), and moreover with different errors at different values of E, the relations obtained in the present work may have advantages over the ordinarily used ones.

The advantages of the new dispersion relations (we have especially in mind (8)) are as follows:

1) They make it possible to use experimental results with the greatest efficiency; for example, using the data from phase analysis $(\operatorname{Re} f(E))$ in the region where they are reliably determined, and the data on the total cross section $(\operatorname{Im} f(E))$ in other energy regions.

2) By varying E_1 and E_2 (or, more generally, a_k, b_k), one can ascertain the possibility of a violation of the dispersion relations due to inaccuracies (errors) associated with the experimental determinations of $\operatorname{Im} f(E)$ and $\operatorname{Re} f(E)$ in different regions of the energy E.

3) The convergence of the integrals as $|E| \to \infty$ in the new dispersion relations is considerably faster than in the ordinarily used ones (because of the factor $\left[\prod_k (E - a_k)(E - b_k)\right]^{-1/2}$). To ensure the same convergence in the ordinary dispersion relations, additional knowledge of the extra-integral terms ([1]) that then appear is required.

The dispersion relation (8) is particularly distinguished by the fact that in it integration over the unphysical region requires only knowledge of $\operatorname{Im} f(E)$, which is known for π-meson–nucleon scattering, while integration over the semi-infinite interval $[E_2, \infty)$ also requires only knowledge of $\operatorname{Im} f(E)$, which for sufficiently large E_2 is well approximated by the assumption that the total cross section $\sigma(E)$ is constant.*

Let us proceed to discussion of the additional relations (6), (7), (9), (10), obtained on the basis of (2), (4). These relations carry essentially new information in comparison with the ordinarily used dispersion relations. Since these additional relations are necessary conditions for the analyticity of $f(E)$ and the finiteness of $f(E)$ at the division points a_k, b_k, while the dispersion relations are necessary and sufficient, it is clear that the additional relations

* In the dispersion relation (5) (see also (8)) it is also necessary to know $\operatorname{Re} f(E)$ as $|E| \to \infty$, which makes this relation of little practical use.

for their verification require a smaller amount of experimental data than dispersion relations.

Relation (6), obviously, makes it possible to estimate the behavior of $\operatorname{Re} f(E)$ in the "nonphysical" region $|E| > \mu$, knowing $\operatorname{Im} f(E)$ in the physical region of energies $E > \mu$, i.e., using only experimental data.

Relation (7) can be used in different ways. First, it makes it possible to estimate the behavior of $\operatorname{Im} f(E)$ in the "nonphysical" region $|E| > \mu$, if it is not known there in advance (for example, in the case of nucleon-nucleon scattering), knowing $\operatorname{Re} f(E)$ in the physical region of energies, i.e., knowing the experimental results. Second, if the imaginary part $\operatorname{Im} f(E)$ in the "nonphysical" region is known (for example, in the case of π-meson–nucleon scattering), then relation (7) makes it possible to obtain restrictions on the behavior of $\operatorname{Re} f(E)$ in the physical region of energies, i.e., to obtain an additional condition which the phase analysis must satisfy. In this same case relation (7) makes it possible, in a way in principle independent of the usual dispersion relations, to determine the constant of the π-meson–nucleon interaction from experimental data, namely only on the basis of phase analysis ($\operatorname{Re} f(E)$). It is important to emphasize that, since, as was already noted, the additional relations (and, in particular, (7)) are necessary (and not necessary and sufficient) conditions of analyticity and finiteness of $f(E)$, determining the interaction constant from them in principle requires less information (a smaller amount of experimental data) concerning the scattering amplitude $f(E)$ than is necessary for determining the interaction constant from dispersion relations.

The additional relations (9), (10), from the principal point of view, are analogous to the relations (6), (7) already considered. Their most important practical advantage is that in them, everywhere on the semi-infinite interval $[E_2, \infty)$, $\operatorname{Im} f(E)$ is integrated, which, in contrast to $\operatorname{Re} f(E)$, at infinitely large energies (for sufficiently large E_2) is well approximated by the assumption of constancy of the total cross section $\sigma(E)$.

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Note: Figure translations are in progress. See original paper for figures.