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Abstract

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ON A CLASS OF PARABOLIC SYSTEMS

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Many works have been devoted to the theory of systems parabolic in the sense of I. G. Petrovsky [(1)], in which the Cauchy problem for linear and nonlinear systems and various properties of solutions have been studied quite fully. The definition of I. G. Petrovsky is based on the assumption of a special construction of the system: differentiation with respect to t is assigned weight p, so that one differentiation with respect to t corresponds to p-fold differentiation with respect to the spatial coordinates $x_1, x_2, \dots, x_n$, while the spatial coordinates are completely equivalent (in the sense of differentiation).

In the present note we define parabolic systems for which the derivatives with respect to each of the spatial coordinates have their own highest order; when these orders coincide, the systems become the usual parabolic systems in the sense of I. G. Petrovsky. For such parabolic systems (we shall call them $2b$-parabolic) fundamental matrices of solutions (f.m.s.) are constructed, and with their aid the correct solvability of the Cauchy problem and the hypoellipticity of such systems are established.

The facts given below are established by means of a method developed in recent years by the author in the study of systems parabolic in the sense of I. G. Petrovsky [(3)], without overcoming any substantial additional difficulties.

1. Consider the system of differential equations

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = \sum_{j=1}^{N} \sum_{k_0 + \frac{k_1}{2b_1} + \dots + \frac{k_n}{2b_n} \le n_j} A_{ij}^{(k_0,k)}(t, x_1, \dots, x_n) \frac{\partial^{k_0+k_1+\dots+k_n} u_j}{\partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}}, \qquad (1)$$

$$i = 1, 2, \dots, N.$$

Definition. We shall call system (1) $2b$-parabolic in a domain G of the space $(x_1, x_2, \dots, x_n, t)$, if for any $(x, t) \in G$ and real $\sigma_1, \sigma_2, \dots, \sigma_n$, $\sigma_1^{2b_1} + \sigma_2^{2b_2} + \dots + \sigma_n^{2b_n} = 1$, the equation

$$\det\left\{\left\|\sum_{k_0+\frac{k_1}{2b_1}+\cdots+\frac{k_n}{2b_n}=n_j} A_{ij}^{(k_0,k)}(t,x)\lambda^{k_0}(i\sigma)^k\right\| - \left\|\begin{matrix}\lambda^{n_1} & & \\ & \ddots & \\ & & \lambda^{n_N}\end{matrix}\right\|\right\} = 0 \quad (2)$$

has roots $\lambda_i(t,x,\sigma)$, the real parts of which satisfy the inequalities $\operatorname{Re}\lambda_i(t,x,\sigma) < -\delta;\ \delta > 0;\ 2b = (2b_1, 2b_2, \dots, 2b_n)$.

To shorten the notation, in what follows we shall formulate all results for system (1) in which $n_1 = n_2 = \cdots = n_N = 1$.

Theorem 1. Suppose: 1) the coefficients of (1) are continuous in t, and the continuity of those coefficients for which

$$\frac{k_1}{2b_1} + \cdots + \frac{k_n}{2b_n} = 1$$

is uniform in x in

$$\Pi_1\{-\infty < x_s < \infty,\ s = 1, 2, \dots, n;\ 0 \le t \le T\};$$

2) the coefficients of (1) are bounded and Hölder-continuous in x.

Then for system (1) there exists in $\Pi_1,\ t > \tau$, an f.m.s. $Z(t,\tau,x,\xi)$ satisfying the inequalities

$$\left|D_x^k Z(t,\tau,x,\xi)\right| \le C(t-\tau)^{-\tilde{k}-\tilde{n}} e^{-c\rho}; \quad (3)$$

$$\left|\Delta_n D_x^k Z(t,\tau,x,\xi)\right| \le C^* \sum_{m=1}^{n} |h_m|^{\alpha_m} (t-\tau)^{-\tilde{k}-\tilde{n}-\frac{\alpha_m}{2b_m}} e^{-c^*\rho}, \quad (4)$$

where

$$D_x^k = \frac{\partial^{|k|}}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_n^{k_n}}; \qquad \rho = \sum_{s=1}^{n} |x_s - \xi_s|^{q_s} (t-\tau)^{-\frac{1}{2b_s-1}}; \qquad q_j = \frac{2b_j}{2b_j - 1};$$

$$x = (x_1, \dots, x_n), \qquad z = x + iy; \qquad \zeta = \xi + iv; \qquad f(x+h) - f(x) = \Delta_h f;$$

$$\frac{k_1}{2b_1} + \frac{k_2}{2b_2} + \cdots + \frac{k_n}{2b_n} = \tilde{k}; \qquad \frac{1}{2b_1} + \frac{1}{2b_2} + \cdots + \frac{1}{2b_n} = \tilde{n};$$

$$\sigma^k = \sigma_1^{k_1}\sigma_2^{k_2}\cdots\sigma_n^{k_n};$$

$$\tilde{k} \le 1; \quad |h_m| \le a(t-\tau)^{\frac{1}{2b_m}}; \quad a \text{ is an arbitrary positive number;}$$

C, c are positive numbers depending only on T; C^*, c^* depend on T and a; $0 < \alpha \le 1$.

If, in addition to condition (1), the following condition is fulfilled:

3) $A_j^{(k_0 k)}(t,x)$ have $k_1 + \cdots + k_n,\ k_0 = 0$, derivatives continuous in the Hölder sense in x with respect to $x_1, x_2, \dots, x_n$,

then $Z'(t, \tau, x, \xi)$, as a function of τ, ξ, is the f.m.s. of the system adjoint to (1).

If, in addition to conditions 1), 2), the following condition is fulfilled:

4) the coefficients of (1) are defined in the domain

$$G_1\{|z_1 - x_1^0| < p,\ -\infty < x_s < \infty,\ s = 2, \dots, n,\ t_1 \leq t \leq t_2\}$$

and are, in it, analytic functions of z_1, bounded and continuous in $x_1, x_2, \dots, x_n, t$, while the continuity in t of the coefficients for which $\tilde{k} = 1$ is uniform in $(z_1, x_2, \dots, x_n)$ from G_1 (it is also assumed that in G_1 system (1) is parabolic; $t < t_2 \leq T$),

then the f.m.s. can be continued into a complex domain in z_1, ζ_1, G_2, in such a way that there it is analytic in the variables z_1, ζ_1. In this case the estimates

$$\left|D_x^k Z(t, \tau, x, \xi)\right| \leq C_0 (t-\tau)^{-\tilde{k}-\tilde{n}} \exp\left\{-c_0 \rho + b_0 |y_1 - \upsilon_1|^{q_1} (t-\tau)^{-\frac{1}{2b_1 - 1}}\right\}. \quad (5)$$

In establishing Theorem 1, the method developed in ([3]) is used, and also the following property of the generalized homogeneity of the roots of equation (2):

$$\lambda(\sigma_1 \rho^{1/2b_1},\ \sigma_2 \rho^{1/2b_2}, \dots, \sigma_n \rho^{1/2b_n}) = \rho \lambda(\sigma_1, \sigma_2, \dots, \sigma_n).$$

Under the same assumptions on the smoothness of the coefficients, one can construct an f.m.s. for finite domains.

2. Theorem 1 and the usual properties of the f.m.s. make it possible to extend to $2b$-parabolic systems results known for systems parabolic in the sense of I. G. Petrovsky ([3]). The solvability of the Cauchy problem is based on the consideration of the domain of definition and values of the integral operator

$$U(x, t) = Au = \int Z(t, \tau, x, \xi) u(\xi, \tau)\, d\xi. \quad (6)$$

Definition. A function $u(x, t)$ belongs to the space $L_{p,k(t),2b}$, $1 \leq p < \infty$, if the p-th power of the function $|u(x, t)|$

$$\times \exp\left\{-\sum_{j=1}^{n} k_j(t) |x_j|^{q_j}\right\},$$

is summable, and

$$\|u(x,t)\|_{L_{p,k(t),2b}} = \left[\int |u(x,t)|^p \exp\left\{-p \sum_{j=1}^{n} k_j(t) |x_j|^{q_j}\right\} dx\right]^{1/p}.$$

A function $u(x, t)$ belongs to the space $L_{\infty,k(t),2b}$ if

$$u(x,t)\exp\left\{-\sum_{j=1}^{n}k_j(t)x_j^{q_j}\right\}$$

is measurable and essentially bounded, and

$$\|u(x,t)\|_{L_{\infty,k(t),2b}} = \sup_x \operatorname{vrai}\left[|u(x,t)|\exp\left\{-\sum_{j=1}^{n}k_j(t)|x_j|^{q_j}\right\}\right].$$

By $L_{p,k(t),2b,s}$ we denote the space of vector-functions with s components, each of which belongs to $L_{p,k(t),2b}$,

$$\|u(x,t)\|_{L_{p,k(t),2b,s}} = \sum_{m=1}^{s}\|u_m\|_{L_{p,k(t),2b}}.$$

Theorem 2. If

$$u(x,t)\in L_{p,k(t),2b,N},$$

where

$$1\le p<\infty; \qquad k_j(t,a) = \frac{(c-\varepsilon)a}{\left[(c-\varepsilon)^{2b_j-1} - a^{2b_j-1}(t-t_0)\right]^{\frac{1}{2b_j-1}}},$$

the number c is from inequality (3), $0<\varepsilon<c$; a is any positive number, and for all t from the segment $[t_0,t_1]$,

$$t_1 = \min_i\left(\frac{c-\varepsilon'}{a}\right)^{2b_j-1}, \qquad \varepsilon<\varepsilon'<c,$$

then

$$\|D_x^k U(x,t)\|_{L_{p,k(t),2b,N}} \le C(\varepsilon)(t-\tau)^{-\tilde{k}}\|u(x,t)\|_{L_{p,k(\tau),2b,N}}. \tag{7}$$

The functions $k_j(t,a)$ have the "group property" :

$$k_j(t-\tau,k(\tau,a)) = k_j(t,a).$$

From Theorems 1 and 2 follow various propositions on the existence of a solution and the correctness of the Cauchy problem for system (1), systems close to the linear systems (3), in classes of functions growing like

$$\exp\left\{c\sum_{j=1}^{n}x_j^{q_j}\right\},$$

which generalize the theorems stated in (3).

Let us note that if the order of growth of the solutions is q_j with respect to x_j, then only local theorems are obtained (even for linear systems). Nonlocal theorems for linear systems are obtained if the order of growth is less than the maximal one, or the growth is maximal and the type is minimal.

3. If the smoothness of the coefficients is increased, then the smoothness of the fundamental matrix of solutions will also increase. Hence, in particular, it follows:

Theorem 3. If the coefficients of system (1) are infinitely differentiable functions of $x_1, \dots, x_n, t$ in some domain G of the space $x_1, \dots, x_n, t$, then any regular solution of (1) is an infinitely differentiable function of $x_1, x_2, \dots, x_n, t$.

Since it is easy to establish that every solution of system (1) in the sense of generalized functions (4) is a regular functional generated by a regular solution of (1), it follows from Theorem 3 that system (1) is hypoelliptic.

4. From the analytic continuability of the matrix $Z(t,\tau,x,\xi)$ and the estimates (5) there follows analyticity with respect to the spatial coordinates of the regular solutions of system (1), representable in the form (6), if the coefficients are analytic functions of these coordinates. This result is valid for solutions of systems

$$\frac{\partial u_i}{\partial t} = \sum_{\tilde{k}=1} A_{ij}^{(k)}(x,t)D_x^k u_j + F_i(t,x,u,\dots,D_x^{k'}u_j,\dots), \qquad \tilde{k}' < 1, \tag{8}$$

$$i = 1, 2, \dots, N,$$

if F_i are analytic in all arguments except t, and are representable in the form

$$u = \int Z\varphi\, d\xi + \int_0^t d\tau \int ZF\, d\xi.$$

If $2b_1 = 2b_2 = \dots = 2b_r > 2b_{r+1} \geq 2b_{r+2} \geq \dots \geq 2b_n$, then every regular solution of (1) is an analytic function of $x_1, x_2, \dots, x_r$, if the coefficients are analytic in $x_1, x_2, \dots, x_r$.

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Note: Figure translations are in progress. See original paper for figures.