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Abstract

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ASYMPTOTICS OF THE GREEN' S FUNCTION OF THE CAUCHY PROBLEM FOR PETROVSKY-CORRECT SYSTEMS WITH TWO VARIABLES AS $t \to +0,\ x \to \infty$

(Presented by Academician I. G. Petrovsky, 1 XII 1959)

In the present note we consider the Cauchy problem for Petrovsky-correct equations and systems with two variables t, x, with coefficients depending on t. For the Green' s function of the Cauchy problem its asymptotics is found as $t \to +0,\ x \to \infty$; the asymptotics obtained is also valid as $t \to +0$ with x fixed and as $x \to \infty$ with $t > 0$ fixed.

1. Let us first consider a single equation of order n with constant coefficients:

$$\frac{\partial u}{\partial t} = P\left(i \frac{\partial}{\partial x}\right) u, \tag{1}$$

$$u\big|_{t=0} = u_0(x). \tag{2}$$

We introduce the following terminology:

1°. Equation (1) is parabolic in the sense of Petrovsky if $P(s) = a_0 s^n + \cdots$, $\operatorname{Re} a_0 < 0$, n even.

2°. Equation (1) is properly parabolic in the sense of Shilov if $P(s) = a_0 s^n + \cdots + a_{n-p} s^p + \cdots$, $\operatorname{Re} a_0 = \operatorname{Re} a_1 = \cdots = \operatorname{Re} a_{n-p-1} = 0$, $\operatorname{Re} a_{n-p} < 0$, p even, $n > p \geq 2$.

3°. An equation is properly correct in the sense of Petrovsky if $P(s) = iQ(s)$, where $Q(s)$ is a polynomial with real coefficients.

The results are summarized in Table 1 $(n > 1)$.

Notes to the table

$$A(x,t) = \frac{c_0}{t^{\frac{1}{2(n-1)}} |x|^{\frac{n-2}{2(n-1)}}} \left[1 + O\left(\frac{t^{\frac{1}{n-1}}}{|x|^{\frac{1}{n-1}}}\right)\right].$$

All constants occurring in the table are real; $iO(|x|)$ is a purely imaginary quantity. The class of correctness is understood as follows:
1) for $0 < t \leq T$, the solution $u(x,t)$ is a classical solution of (1);
2) continuous dependence of the solution on the initial data is understood in the sense of (^{2}).

For sufficiently small T, $0 \leq t \leq T$, in cases I, II the solution $u(x,t)$ belongs to the same class with b_0 replaced by any $b_0' > b_0$. In case III, 1°, and in case III, 2°, on the corresponding half-plane, the solution of the Cauchy problem is a function of power growth of order $\leq +l$.

The asymptotics of the derivatives of the Green' s functions is expressed in terms of the asymptotics of the Green' s functions as follows:

$$\frac{\partial^r G}{\partial t^r} \sim C_r \left(\frac{|x|}{t}\right)^{\frac{nr}{n-1}} G(x,t), \tag{3}$$

$$\frac{\partial^r G}{\partial x^r} \sim D_r \left(\frac{|x|}{t}\right)^{\frac{r}{n-1}} G(x,t). \tag{4}$$

	Green' s function	Class of well-posedness	Not the class of well-posedness
Parabolic in the sense of Petrovskii I$\|x\| \to \infty$	$G(x,t) =$ $A(x,t) \times \times$ $\exp\left[-(C_1 + iC_2)\frac{\|x\|^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}}} + O(\|x\|)\right], \quad C_1 > 0$	$\|u_0(x)\| \leq$ $C\exp\left[b_0\|x\|^{\frac{n}{n-1}}\right]$	$\|u_0(x)\| \leq$ $C\exp\left[b_0\|x\|^{\frac{n}{n-1}} + \delta\right], \quad \delta > 0$
Properly parabolic in the sense of Shilov II1°. n even,$\|x\| \to \infty$	$G(x,t) =$ $A(x,t)\exp\left[iC_1\frac{\|x\|^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}}} + iO(\|x\|) - C_2\|x\|^{\frac{p}{n-1}} t^{\frac{n-p-1}{n-1}} + O\left(\|x\|^{\frac{p-1}{n-1}} t^{\frac{n-p}{n-1}}\right)\right], C_2 > 0$	$\|u_0(x)\| \leq$ $C_\varepsilon \exp\left[\varepsilon\|x\|^{\frac{p}{n-1}}\right]$ for any $\varepsilon > 0$	$\|u_0(x)\| \leq$ $C\exp\left[\alpha\|x\|^{\frac{p}{n-1}}\right], \quad \alpha > 0$
Properly parabolic in the sense of Shilov II2°. n odd.a) $x \to +\infty \times \times \operatorname{sign}(\operatorname{Im} a_0)$b) $x \to -\infty \times \times \operatorname{sign}(\operatorname{Im} a_0)$	a) $G(x,t) =$ $A(x,t)\sin\left(C_2\frac{\|x\|^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}}}\right) \times \times$ $\exp\left[-C_2\|x\|^{\frac{p}{n-1}} t^{\frac{n-p-1}{n-1}} + iO(\|x\|) + + O\left(\|x\|^{\frac{p-1}{n-1}} t^{\frac{n-p}{n-1}}\right)\right], \quad C_2 > 0$b) $G(x,t) =$ $\exp\left[-C_1\frac{\|x\|^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}}} + O(\|x\|)\right] \times \times$ $\sin\left(C_3 + C_4\frac{\|x\|^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}}}\right) A(x,t)$,$C_1 \geq 0$	$\|u_0(x)\| \leq$ $C\exp\left[b_0\|x\|^{\frac{n}{n-1}}\right]$sign $x =$ $-\operatorname{sign}(\operatorname{Im} a_0)$$|u_0(x)| \leq$ $C_\varepsilon\exp\left[\varepsilon\|x\|^{\frac{p}{n-1}}\right]$,for any $\varepsilon > 0$,sign $x =$ $\operatorname{sign}(\operatorname{Im} a_0)$	$\|u_0(x)\| \leq$ $C\exp\left[b_0\|x\|^{\frac{n}{n-1}} + \delta\right], \quad \delta > 0$,sign $x =$ $-\operatorname{sign}(\operatorname{Im} a_0)or\|u_0(x)\| \leq$ $C\exp\left[\alpha\|x\|^{\frac{p}{n-1}}\right], \quad \alpha > 0$,sign $x =$ $\operatorname{sign}(\operatorname{Im} a_0)$

III. Properly Petrovsky-correct			
1°. n even, $\|x\| \to \infty$	$G(x,t) = A(x,t)\exp\left[iC_1\frac{\|x\|^{-\frac{n}{n-1}}}{t^{\frac{1}{n-1}}} + iO(\|x\|)\right]$	$\left\|u_0^{(k)}(x)\right\| \ll C_k(\|x\|+1)^l$, $k \geq n + [n/2] + 1$	$\|u_0(x)\| \ll e^{\|x\|^\alpha}$, $\alpha > 0$
2°. n odd. a) $x \to +\infty \times \operatorname{sign}(\operatorname{Im} a_0)$	$G(x,t) = A(x,t)\sin\left(C_1\frac{\|x\|^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}}}\right)$	$\left\|u_0^{(k)}(x)\right\| \ll C_k(\|x\|+1)^l$, $k \geq n + [n/2] + 1$, $\operatorname{sign} x = \operatorname{sign}(\operatorname{Im} a_0)$	The same as in III, 1°, if $\operatorname{sign} x = \operatorname{sign}(\operatorname{Im} a_0)$
2°. n odd. b) $x \to -\infty \times \operatorname{sign}(\operatorname{Im} a_0)$	$G(x,t) = A(x,t)\exp\left[-C_1\frac{\|x\|^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}}} + O(\|x\|)\right]$, $C_1 > 0$	$\|u_0(x)\| \ll C\exp\left[-b_0\|x\|^{\frac{n}{n-1}}\right]$, $\operatorname{sign}(\operatorname{Im} a_0)$	$\|u_0(x)\| \ll C\exp\left[-b\|x\|^{\frac{n}{n-1}+\delta}\right]$, $\operatorname{sign} x = -\operatorname{sign}(\operatorname{Im} a_0)$

Let us also note that the Green function for $t > 0$, with x fixed, is an entire function of x.

The asymptotics of the Green function had previously been unknown, but for equations parabolic in the sense of Petrovsky its exact estimate was known (see, for example, ([1])).

Correctness classes I and II, 1° were known earlier ([2]); for the case II, 2° the same correctness class was obtained as for II, 1°. Thus, in the cases II, 2° and III, 2°, the correctness classes obtained are substantially broader than those known earlier. The question of the maximality of the correctness classes II and III, solved here, had not previously been studied.

The most interesting of the results obtained is that the Green function $G(x,t)$ decreases with different rates on the rays $x > 0$ and $x < 0$. This is not surprising, since $G(x,t)$ is an entire function of x and therefore may behave differently on different rays.

2. Consider the Green function of the Cauchy problem for a Petrovsky-correct equation with constant coefficients, of order $n > 1$ in t, solved with respect to $\partial^n u/\partial t^n$, i.e., the solution of the problem

$$P\left(\frac{\partial}{\partial t}, i\frac{\partial}{\partial x}\right)u = 0, \tag{5}$$

$$u\Big|_{t=0} = \frac{\partial u}{\partial t}\Big|_{t=0} = \cdots = \frac{\partial^{n-2}u}{\partial t^{n-2}}\Big|_{t=0} = 0,$$

$$\left.\frac{\partial^{n-1}u}{\partial t^{n-1}}\right|_{t=0} = \delta(x). \tag{6}$$

Lemma (V. M. Borok (³)). *The characteristic roots of a system with constant coefficients that is Petrovsky-correct, for large s, have one of the following expansions in Puiseux series:*

$$1^\circ. \quad \lambda_j(s) = a_{0j}s^{k_{0j}} + \dots, \quad k_{0j} > 0$$

even, $\operatorname{Re} a_{0j} < 0$*;*

$$2^\circ. \quad \lambda_j(s) = a_{0j}s^{k_{0j}} + \dots + a_{pj}s^{k_{pj}} + \dots,$$

$$k_{0j} > \dots > k_{pj} > 0,$$

the numbers $k_{0j}, k_{1j}, \dots, k_{pj}$ are integers, k_{pj} is even,

$$\operatorname{Re} a_{0j} = \operatorname{Re} a_{1j} = \dots = \operatorname{Re} a_{p-1,j} = 0, \quad \operatorname{Re} a_{pj} < 0;$$

3°. $\lambda_j(s) = a_{0j}s^{k_{0j}} + \dots + a_{1j}s + \dots$, the numbers $k_{0j}, k_{1j}, \dots$ are integers, $k_{0j} \geq 2$, $k_{0j} > k_{1j} > \dots \geq 1$, $\operatorname{Re} a_{0j} = \dots = \operatorname{Re} a_{1j} = 0$.

4°. $\lambda_j(s) = a_{0j}s + a_{1j} + a_{2j}s^{-k_{0j}} + \dots$, $k_{0j} > 0$.

In this case the system cannot have, as $\lambda_j(s)$, only roots of type 4°, with $a_{0j} = 0$.

Here we shall give only the asymptotics of $G(x,t)$; the classes of well-posedness and ill-posedness are obtained from this in a trivial manner.

Denote by α_j the order of growth of $\prod_k' (\lambda_j(s) - \lambda_k(s))$ with respect to s, $\alpha_j \geq 1$.

Theorem 1. As $t \to +0$, $x \to \infty$, the Green' s function is equal to the sum of n terms. Each root of type 1°, 2°, 3° gives a term of the same kind as in the table, with n replaced by k_{0j}, p by k_{pj}, multiplied, moreover, by

$$B_j(|x|/t)^{-\frac{\alpha_j}{k_{0j}-1}}.$$

When differentiating with respect to t, the term corresponding to the root $\lambda_j(s)$ is multiplied by $(|x|/t)^{\frac{k_{0j}}{k_{0j}-1}}$; when differentiating with respect to x, by $(|x|/t)^{\frac{1}{k_{0j}-1}}$. The terms corresponding to roots of type 4°, and their derivatives with respect to x and t up to order $\alpha_j - 1$ inclusive, are finite functions. Derivatives with respect to x of order $m \geq \alpha_j$ and derivatives with respect to t, if $a_{0j} \neq 0$, are sums of a finite number of derivatives of the δ-function of order $m - \alpha_j$. If $a_{0j} = 0$, then all derivatives with respect to t are finite.

Remark 1. $G(x,t)$ is always a function. If equation (5) has characteristic roots only of types $1^\circ, 2^\circ$, and 3°, then $G(x,t)$ is an entire function of x for every $t > 0$.

Remark 2. Every solution of a system of equations with constant coefficients that is well-posed in the sense of Petrovsky is obtained from some solution of an equation of the form (5) by applying to it a differential operator $T(i\partial/\partial x)$ with constant coefficients [4]. If the system has characteristic roots only of types 1° and 2°, then the classes of well-posedness of the equation and of the system coincide; if there are also roots of type 3° or 4°, then the initial data are required to have smoothness higher by a finite order and boundedness of several higher derivatives than in the equation.

Remark 3. If the coefficients of the equation depend on t, then all the results stated above are valid, provided that the previously assumed conditions on the coefficients $P(s)$ are required to hold uniformly in t for $0 \leq t \leq T$.

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Note: Figure translations are in progress. See original paper for figures.

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